On the Numerical Solution of Equilibria in Auction Models with Asymmetries within the Private-Values Paradigm

Timothy P. Hubbard[a], Harry J. Paarsch[b,*]

[a] *Department of Economics, Texas Tech University, USA*
[b] *Amazon.com, Inc., USA*

Abstract

We present a survey of numerical methods used to approximate equilibrium bid functions in models of auctions as games of incomplete information where private values are modelled as draws from bidder-specific type distributions when pay-your-bid rules are used to determine transactions prices. We provide a formal comparison of the performance of these numerical methods (based on speed and accuracy) and suggest ways in which they can be improved and extended as well as applied to new settings.

Key words: first-price auctions; asymmetric auctions; private-values models; numerical solutions.

1. Motivation and Introduction

During the past half century, economists have made considerable progress in understanding the theoretical structure of strategic behavior under market mechanisms, such as auctions, when the number of potential participants is relatively small; see Krishna [34] for a comprehensive presentation and evaluation of progress.

Perhaps the most significant breakthrough in understanding behavior at auctions was made by Vickrey [59] who modelled auctions as non-cooperative games of incomplete information where bidders have private information concerning their type that they exploit when tendering offers for the good for sale. One analytic device commonly used to describe bidder motivation at auctions is a continuous random variable that represents individual-specific heterogeneity in types, which is typically interpreted as heterogeneity in valuations. The conceptual experiment involves each potential bidder's receiving an independent draw from a distribution of valuations. Conditional on his draw, a bidder is assumed to act purposefully, maximizing either the expected profit or the expected utility of profit from winning the good for sale. Another frequently-made assumption is that the bidders are *ex ante* symmetric, their independent draws coming from the same distribution of valuations, an assumption that then allows the researcher to focus on a representative agent's decision rule when characterizing the Bayes–Nash equilibrium to the

*Corresponding author: Amazon.com; P.O. Box 81266; Seattle, WA 98108-1300, USA.
Email addresses: Timothy.Hubbard@ttu.edu (Timothy P. Hubbard), HPaarsch@Gmail.com (Harry J. Paarsch)

auction game, particularly under pay-your-bid pricing rules, which are referred to as *first-price auctions*, at least by economists.[1]

The assumption of symmetry is made largely for computational convenience. When the draws of potential bidders are independent, but from different distributions (urns, if you like), then the system of first-order differential equations that characterizes a Bayes–Nash equilibrium usually does not have a convenient closed-form solution: typically, approximate solutions can only be calculated numerically.[2]

Of course, admitting several objects complicates matters considerably as the research of Weber [60], for example, has shown. In fact, economic theorists distinguish between multi-object and multi-unit auctions. At multi-unit auctions, it matters not which unit a bidder wins, but rather the aggregate number of units he wins, while at multi-object auctions it matters which specific object(s) a bidder wins. An example of a multi-object auction would involve the sale of an apple and an orange, while an example of a multi-unit auction would involve the sale of two identical apples. At the heart of characterizing Bayes–Nash equilibria in private-values models of sequential, multi-unit auctions (under either first-price or second-price rules) is the solution to an asymmetric-auction game of incomplete information. In addition, an asymmetric first-price model is relevant even when bidders are assumed to draw valuations from the same distribution, but have different preferences (for example, risk-averse bidders might differ by their Arrow-Pratt coefficient of relative risk aversion), when bidders collude and/or form coalitions, and also when the auctioneer (perhaps the government) grants preference to a class of bidders. Bid preferences are particularly interesting because the auctioneer, for whatever reason, deliberately introduces an asymmetry when evaluating bids, even though bidders may be symmetric. Thus, understanding how to solve for equilibria in models of asymmetric auctions is of central importance to economic theory as well as to empirical analysis and policy evaluation.

Computation time is of critical importance to structural econometricians who often need to solve for the equilibrium (inverse-) bid functions within an estimation routine for each candidate vector of parameters when recovering the distributions of the latent characteristics, which may be conditioned on covariates as well.[3] Most structural econometric work is motivated by the fact that researchers would like to consider counterfactual exercises that allow for policy recommendations. Because users of auctions are interested in raising as much revenue as possible (or, in the case of procurement, saving as much money as possible), the design of an optimal auction is critical to raising (or saving) money. If applied researchers can capture reality sufficiently well in their models, then they can influence policies at auctions in practice.[4] Unfortunately poor approximations to the bidding strategies can lead to biased and inconsistent estimates of the structural elements of the model. Thus, both speed and accuracy are important considerations when solving asymmetric first-price auctions. It will be important to keep both of these considerations in mind as we investigate ways in which to solve such models.

Our chapter is in six additional sections: in the next section, we first develop a notation that is

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[1]Within the private-values environment, under the pricing rules used at oral, ascending-price (English) or second-price, sealed-bid (Vickrey) formats, it is a dominant strategy for a bidder to tender his valuation, so computing the equilibrium is relatively straightforward.

[2]Within the private-values environment, under the second-price rule, it remains a dominant strategy for a bidder to tender his valuation, so computing the equilibrium remains straightforward.

[3]Likewise, if researchers need to simulate dynamic games that require computing the inverse-bid functions in each period, then speed is crucial because this may require solving for the inverse-bid functions thousands of times.

[4]For example, the research reported by Paarsch [48] determined the reserve prices set at timber auctions conducted by the British Columbian Ministry of Forests in the 1990s and beyond.

used in the remainder of the chapter, then introduce some known results, and finally demonstrate how things work within a well-understood environment. We apologize in advance for abusing the English language somewhat: specifically, when we refer to a first-price auction, we mean an auction at which either the highest bidder wins the auction and pays his bid or, in a procurement context, the lowest bidder wins the auction and is paid his bid. This vocabulary is standard among researchers concerned with auctions; see, for example, Paarsch and Hong [49]. When the distinction is important, we shall be specific concerning what we mean.

Subsequently, in section 3, we describe some well-known numerical strategies that have been used to solve two-point boundary-value problems that are similar to ones researchers face when investigating asymmetric first-price auctions. We use this section not just as a way of introducing the strategies, but so we can refer to them later when discussing what researchers concerned with solving for equilibrium (inverse-) bid functions at asymmetric first-price auctions have done. In section 4, we then discuss research that either directly or indirectly contributed to improving computational strategies to solve for bidding strategies at asymmetric first-price auctions. Specifically, we describe in detail the work of Marshall, Meurer, Richard, and Stromquist [41]; Bajari [3]; Fibich and Gavious [14]; Li and Riley [37]; Gayle and Richard [18]; Hubbard and Paarsch [23]; Fibich and Gavish [15] as well as Hubbard, Kirkegaard, and Paarsch [26]. In section 5, we depict the solutions to some examples of asymmetric first-price auctions to illustrate how the numerical stratgies can be used to investigate problems that would be difficult to analyze analytically. In fact, following the work of Hubbard and Paarsch [24], we present one example that has received very little attention thus far—asymmetric auctions within the affiliated private-values paradigm (APVP). In section 6, we compare and contrast the established strategies and suggest ways in which they can be extended or improved by additional future research. We summarize and conclude in section 7. Note, too, that we provide the computer code used to solve the examples of asymmetric first-price auctions presented below at the following website:

`http://www.myweb.ttu.edu/timhubba/code/hpfpacode.zip`

2. General Model

In this section, we first develop a notation, then introduce some known results, and finally demonstrate how our methods work within a well-understood environment.

2.1. Notation

Consider a set $\mathcal{N}$ which equals $\{1, 2, \ldots, N\}$, and let the letter n index the members of the set who are potential bidders at the auction. Because the main focus in auction theory is asymmetric information, which economic theorists have chosen to represent as random variables, the bulk of our notation centers around a consistent way to describe random variables. Typically, we denote random variables by uppercase roman letters—e.g., V or C. Realizations of random variables are then denoted by lowercase roman letters; for example, v is a realization of V. Probability density and cumulative distribution functions are denoted f and F, respectively, where the trailing subscript denotes a class of bidder—e.g., $F_0(v)$. When there are different distributions (urns), we again use the subscript to refer to a given bidder's distribution, but use the set $\mathcal{N}$ numbering. Hence, f_1 as well as F_N, but f_n and F_n, in general. If necessary, a vector $(V_1, V_2, \ldots, V_N)$ of random variables is denoted $\boldsymbol{V}$, while a realization, without bidder n is denoted $\boldsymbol{v}_{-n}$. The vectors $(f_1, f_2, \ldots, f_N)$ and $(F_1, F_2, \ldots, F_N)$ are denoted $\boldsymbol{f}$ and $\boldsymbol{F}$.

The lowercase Greek letters β and σ are used to denote equilibrium bid functions: σ for a bid at a first-price auction where the choice variable is s. Again, if necessary, we use $\boldsymbol{\sigma}$ to collect all strategies, and $\boldsymbol{s}$ to collect the choice variables. Also, $\boldsymbol{\sigma}_{-n}$ is used to collect all strategies minus bidder n, while $\boldsymbol{s}_{-n}$ collects all the choices minus bidder n. We use φ to denote the inverse-bid function and $\boldsymbol{\varphi}$ to collect all of the inverse-bid functions. Now, β denotes a tender at a low-price auction, where the choice variable is b. We use $\boldsymbol{\beta}$ to collect all strategies and $\boldsymbol{b}$ to collect the choice variables. $\boldsymbol{\beta}_{-n}$ is used to collect all strategies minus bidder n, while $\boldsymbol{b}_{-n}$ collects all the choices minus bidder n.

We denote by $\mathbb{P}$ a general family of polynomials, and use $\mathbb{T}$ for Chebyshev polynomials, and $\mathbb{B}$ for Bernstein polynomials.

We use $\boldsymbol{\alpha}$ to collect the parameters of the approximate equilibrium inverse-bid functions.

2.2. Derivation of Symmetric Bayes–Nash Equilibrium

Consider a seller who seeks to divest a single object at the highest price. The seller invites sealed-bid tenders from N potential buyers. After the close of tenders, the bids are opened more or less simultaneously and the object is awarded to the highest bidder. The winner then pays the seller what he bid.

Suppose each potential buyer has a private value for the object for sale. Assume that each potential buyer knows his private value, but not those of his competitors. Assume that V_n, the value of potential buyer n, is an independent draw from the cumulative distribution function $F_0(v)$, which is continuous, having an associated probability density function $f_0(v)$ that is positive on the compact interval $[\underline{v}, \overline{v}]$ where $\underline{v}$ is weakly greater than zero. Assume that the number of potential buyers N as well as the cumulative distribution function of values $F_0(v)$ and the support $[\underline{v}, \overline{v}]$ are common knowledge. This environment is often referred to as the symmetric *independent private-value paradigm* (IPVP).

Suppose potential buyers are risk neutral. Thus, when buyer n, who has valuation v_n, submits bid s_n, he receives the following payoff:

$$\mathsf{Payoff}(v_n, s_n) = \begin{cases} v_n - s_n & \text{if } s_n > s_m \text{ for all } n \neq m \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Assume that buyer n chooses s_n to maximize his expected profit

$$U_n(s_n) = (v_n - s_n) \Pr(\mathsf{win}|s_n). \tag{2}$$

What is the structure of $\Pr(\mathsf{win}|s_n)$? Within this framework, the identity of bidders (their subscript n) is irrelevant because all bidders are *ex ante* identical. Thus, without loss of generality, we can focus on the problem faced by bidder n. Suppose the opponents of bidder n are using a monotonically, increasing function $\sigma(v)$ to bid. Bidder n will win the auction with tender s_n when all of his opponents bid less than him because their valuations of the object are less than his. Thus,

$$\begin{aligned}
\text{Pr(win}|s_n) &= \Pr(S_1 < s_n, S_2 < s_n, \ldots, S_{n-1} < s_n, S_{n+1} < s_n, \ldots, S_N < s_n) \\
&= \Pr\left[(S_1 < s_n) \cap (S_2 < s_n) \cap \ldots \cap (S_{n-1} < s_n) \cap (S_{n+1} < s_n) \cap \ldots \cap (S_N < s_n)\right] \\
&= \prod_{m \neq n} \Pr\left(S_m < s_n\right) \\
&= \prod_{m \neq n} \Pr\left[\sigma(V_m) < s_n\right] \\
&= \prod_{m \neq n} \Pr\left[V_m < \sigma^{-1}(s_n)\right] \\
&= F_0\left[\sigma^{-1}(s_n)\right]^{N-1} \\
&\equiv F_0\left[\varphi(s_n)\right]^{N-1},
\end{aligned}$$

so equation (2) can be written as

$$U_n(s_n) = (v_n - s_n)\,\text{Pr(win}|s_n) = (v_n - s_n)F_0\left[\varphi(s_n)\right]^{N-1} \tag{3}$$

where $\varphi(\cdot)$ is the inverse-bid function. Differentiating equation (3) with respect to s_n yields the following first-order condition:

$$\begin{aligned}
\frac{\mathrm{d}U_n(s_n)}{\mathrm{d}s_n} = -F_0\left[\varphi(s_n)\right]^{N-1} + \\
(v_n - s_n)\,(N-1)F_0\left[\varphi(s_n)\right]^{N-2} f_0\left[\varphi(s_n)\right] \frac{\mathrm{d}\varphi(s_n)}{\mathrm{d}s_n} = 0.
\end{aligned} \tag{4}$$

In a Bayes–Nash equilibrium, $\varphi(s)$ equals v. Also, under monotonicity, we know that $\mathrm{d}\sigma(v)/\mathrm{d}v$ equals $\mathrm{d}s/\mathrm{d}\varphi(s)$, so dropping the n subscript yields

$$\frac{\mathrm{d}\sigma(v)}{\mathrm{d}v} + \sigma(v)\frac{(N-1)f_0(v)}{F_0(v)} = \frac{(N-1)vf_0(v)}{F_0(v)}. \tag{5}$$

Note that, within the symmetric IPVP, optimal behavior is characterized by a first-order ordinary differential equation (ODE); that is, the differential equation involves only the valuation v, the bid function $\sigma(v)$, and the first derivative of the bid function $\mathrm{d}\sigma(v)/\mathrm{d}v$, which we shall often denote in a short-hand by $\sigma'(v)$, below. Although the valuation v enters nonlinearly through the functions $f_0(v)$ and $F_0(v)$, the differential equation is considered linear because $\sigma'(v)$ can be expressed as a linear function of $\sigma(v)$. These features make the solution to this differential equation tractable, but as we shall see in the subsection that follows, they only hold within the symmetric IPVP.

Equation (5) is among the few differential equations that have a closed-form solution. Following Boyce and DiPrima [5] and using a notation that will be familiar to students of calculus, we note that when differential equations are of the following form:

$$y' + p(x)y = q(x)$$

there exists a function $\mu(x)$ such that

$$\mu(x)[y' + p(x)y] = [\mu(x)y]' = \mu(x)y' + \mu'(x)y.$$

Thus,

$$\mu(x)p(x)y = \mu'(x)y.$$

When μ is positive, as it will be in the auction case because it is the ratio of two positive functions multiplied by a positive integer,

$$\frac{\mu'(x)}{\mu(x)} = p(x),$$

so

$$\log[\mu(x)] = \int_{x_0}^{x} p(u)\,\mathrm{d}u,$$

whence

$$\mu(x) = \exp\left[\int_{x_0}^{x} p(u)\,\mathrm{d}u\right].$$

Therefore,

$$\mu(x)y = \int_{x_0}^{x} \mu(u)q(u)\,\mathrm{d}u + k$$

for some constant k, or

$$y = \frac{1}{\mu(x)}\left[\int_{x_0}^{x} \mu(u)q(u)\,\mathrm{d}u + k\right]$$

where k is chosen to satisfy an initial condition $y(x_0)$ equals y_0.

To solve equation (5), one must impose a boundary condition. In the absence of a reserve price, a minimum price that must be bid, the condition typically imposed is $\sigma(\underline{v})$ equals $\underline{v}$. In words, a potential buyer having the lowest value $\underline{v}$ will bid his value. In the presence of a reserve price r_0, one has $\sigma(r_0)$ equals r_0. The appropriate initial condition, together with the differential equation, constitute an initial-value problem which has the following unique solution:

$$\sigma(v) = v - \frac{\int_{r_0}^{v} F_0(u)^{N-1}\,\mathrm{d}u}{F_0(v)^{N-1}}. \tag{6}$$

This is the symmetric Bayes–Nash equilibrium bid function of the n^{th} bidder; it was characterized by Holt [22] as well as Riley and Samuelson [53]. Before continuing on to the case where bidders are *ex ante* asymmetric, we first introduce some techniques used in the numerical analysis of first-order ODEs which might be used to solve such an initial-value problem.

2.3. *Numerical Solution of First-Order ODEs*

Consider the following first-order ODE for σ as a function of v:

$$\frac{\mathrm{d}\sigma(v)}{\mathrm{d}v} = D(v, \sigma). \tag{7}$$

Several different numerical methods exist to solve differential equations like (7). The simplest of the finite difference methods is, of course, *Euler's method*: starting at v_0, an initial v—say, $\underline{v}$, where $\sigma(\underline{v})$ is $\underline{v}$ in this case—the value of $\sigma(\underline{v}+h)$ can then be approximated by the value of $\sigma(\underline{v})$ plus the step h multiplied by the slope of the function, which is the derivative of $\sigma(v)$, evaluated at $\underline{v}$. This is simply a first-order Taylor-series expansion, so

$$\sigma(\underline{v} + h) \approx \sigma(\underline{v}) + h\left.\frac{\mathrm{d}\sigma(v)}{\mathrm{d}v}\right|_{v=\underline{v}} = \sigma(\underline{v}) + hD[\underline{v}, \sigma(\underline{v})].$$

Denoting this approximate value by σ_1, and the initial value by σ_0, we have

$$\sigma_1 = \sigma(\underline{v}) + h\frac{\mathrm{d}\sigma(v)}{\mathrm{d}v}\bigg|_{v=\underline{v}} = \sigma(\underline{v}) + hD[\underline{v}, \sigma(\underline{v})] = \sigma_0 + hD(v_0, \sigma_0) = \sigma_0 + hD_0. \tag{8}$$

If one can calculate the value of $\mathrm{d}\sigma/\mathrm{d}v$ at $\underline{v}$ using equation (7), then one can generate an approximation for the value of σ at v equal $(\underline{v} + h)$ using equation (8). One can then use this new value of σ, at $(\underline{v} + h)$, to find $\mathrm{d}\sigma/\mathrm{d}v$ (at the new v) and repeat. When $D(v, \sigma)$ does not change too quickly, the method can generate an approximate solution of reasonable accuracy. For example, on an infinite-precision computer, the local truncation error is $O(h^2)$, while the global error is $O(h)$—first-order accuracy.

When the differential equation changes very quickly in response to a small step h, then it is often referred to as a *stiff* differential equation. To solve stiff differential equations accurately using Euler's method, h must be very small, which means that Euler's methods will take a long time to compute an accurate solution. While this may not be an issue when one just wants to do this once, in empirical work concerning auctions, one may need to solve the differential equation thousands (even millions) of times.

Perhaps the most well-known generalization of Euler's method is a family of methods referred to collectively as *Runge–Kutta methods*. Of all the members in this family, the one most commonly used is the fourth-order method, sometimes referred to as *RK4*. Under RK4,

$$\sigma_{k+1} = \sigma_k + h\frac{1}{6}(d_1 + 2d_2 + 2d_3 + d_4)$$

where

$$\begin{aligned}
d_1 &= D\left(v_k, \sigma_k\right) \\
d_2 &= D\left(v_k + \frac{1}{2}h, \sigma_k + \frac{1}{2}hd_1\right) \\
d_3 &= D\left(v_k + \frac{1}{2}h, \sigma_k + \frac{1}{2}hd_2\right) \\
d_4 &= D\left(v_k + h, \sigma_k + hd_3\right).
\end{aligned}$$

Thus, the next value σ_{k+1} is determined by the current one σ_k, plus the product of the step size h and an estimated slope. The estimated slope is a weighted average of slopes: d_1 is the slope at the left endpoint of the interval; d_2 is the slope at the midpoint of the interval, using Euler's method along with slope d_1 to determine the value of σ at the point $(v_k + \frac{1}{2}h)$; d_3 is again the slope at the midpoint, but now the slope d_2 is used to determine σ; and d_4 is the slope at the right endpoint of the interval, with its σ value determined using d_3. Assuming the Lipschitz condition is satisfied, the local truncation error of the RK4 method is $O(h^5)$, while the global truncation error is $O(h^4)$, which is an huge improvement over Euler's method. Note, too, that if $D(\cdot)$ does not depend on σ, so the differential equation is equivalent to a simple integral, then RK4 is simply *Simpson's rule*, a well-known and commonly-used quadrature rule.[5]

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[5] Consider the following integral:

$$F(\ell, u) = \int_{\ell}^{u} f(x)\, \mathrm{d}x.$$

Like Euler's method, however, Runge–Kutta methods do not always perform well on stiff problems; for more on this, see Hairer and Wanner [21]. Note, too, that neither the method of Euler nor the methods of Runge–Kutta use past information to improve the approximation as one works to the right.

In response to these limitations, numerical analysts have pursued a variety of other strategies. For a given h, these alternative methods are more accurate than Euler's method, and may have a smaller error constant than Runge–Kutta methods as well. Some of the alternative methods are referred to as *multi-step methods*. Under multi-step methods, one again starts from an initial point $\underline{v}$ and then takes a small step h forward in v to find the next value of σ. The difference is that, unlike Euler's method (which is a single-step method that refers only to one previous point and its derivative at that point to determine the next value), multi-step methods use some intermediate points to obtain an higher-order approximation of the next value. Multi-step methods gain efficiency by keeping track of as well as using the information from previous steps rather than discarding it. Specifically, multi-step methods use the values of the function at several previous points as well as the derivatives (or some of them) at those points.

Linear multi-step methods are special cases in the class of multi-step methods. As the name suggests, under these methods, a linear combination of previous points and derivative values is used to approximate the solution. Denote by m the number of previous steps used to calculate the next value. Denote the desired value at the current stage by σ_{k+m}. A linear multi-step method has the following general form:

$$\begin{aligned}\sigma_{k+m} &+ \lambda_{m-1}\sigma_{k+m-1} + \lambda_{m-2}\sigma_{k+m-2} + \cdots + \lambda_0\sigma_k \\ &= h\left[\kappa_m D(v_{k+m}, \sigma_{k+m}) + \kappa_{m-1} D(v_{k+m-1}, \sigma_{k+m-1}) + \cdots + \kappa_0 D(v_k, \sigma_k)\right].\end{aligned}$$

The values chosen for $\lambda_0, \ldots, \lambda_{m-1}$ and $\kappa_0, \ldots, \kappa_m$ determine the solution method; a numerical analyst must choose these coefficients. Often, many of the coefficients are set to zero. Sometimes, the numerical analyst chooses the coefficients so they will interpolate $\sigma(v)$ exactly when $\sigma(v)$ is a k^{th} order polynomial. When κ_m is nonzero, the value of σ_{k+m} depends on the value of $D(v_{k+m}, \sigma_{k+m})$, and the equation for σ_{k+m} must be solved iteratively, using fixed-point iteration or, alternatively, using variants of the method of Newton–Raphson.

A simple linear, multi-step method is the *Adams–Bashforth two-step method*. Under this method,

$$\sigma_{k+2} = \sigma_{k+1} + h\frac{3}{2}D(v_{k+1}, \sigma_{k+1}) - h\frac{1}{2}D(v_k, \sigma_k).$$

To wit, λ_1 is -1, while κ_2 is zero, and κ_1 is $\frac{3}{2}$, while κ_0 is $-\frac{1}{2}$. However, to implement Adams–Bashforth, one needs two values (σ_{k+1} and σ_k) to compute the next value σ_{k+2}. In a typical

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To implement Simpson's rule, take the interval $[\ell, u]$ and subdivide it into an even number T of subintervals each of width h, so $[\ell, \ell + h]$, $[\ell + h, \ell + 2h]$, …, $[u - h, u]$. Replace $f(x)$ by a quadratic polynomial that takes the same values as the integrand at the end points of each subinterval and at each midpoint. On any subinterval, $[x_t, x_{t+1}]$,

$$\int_{x_t}^{x_{t+1}} f(x)\,\mathrm{d}x \approx \frac{h}{6}\left[f(x_t) + 4f\left(\frac{x_t + x_{t+1}}{2}\right) + f(x_{t+1})\right].$$

For the entire interval $[\ell, u]$, the formula is

$$\int_{\ell}^{u} f(x)\,\mathrm{d}x \approx \frac{h}{3}\Big[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots + 4f(x_{T-1}) + f(x_T)\Big].$$

initial-value problem, only one value is provided; in the case of auctions, for example, $\sigma(\underline{v})$ or σ_0 equals $\underline{v}$ or v_0 is the only condition provided. One way to circumvent this lack of information is to use the σ_1 computed by Euler's method as the second value. With this choice, the Adams–Bashforth two-step method yields a candidate approximating solution.

For other values of m, Butcher [8] has provided explicit formulas to implement the Adams–Bashforth methods. Again, assuming the Lipschitz condition is satisfied, the local truncation error of the Adams–Bashforth two-step method is $O(h^3)$, while the global truncation error is $O(h^2)$. (Other Adams–Bashforth methods have local truncation errors that are $O(h^5)$ and global truncation errors that are $O(h^4)$, and are, thus, competitive with RK4.)

In addition to Adams–Bashforth, two other families are also used: first, Adams–Moulton methods and, second, backward differentiation formulas (BDFs).

Like Adams–Bashforth methods, the Adams–Moulton methods have λ_{m-1} equal -1 and the other λ_is equal to zero. However, where Adams–Bashforth methods are explicit, Adams–Moulton methods are implicit. For example, when m is zero, under Adams–Moulton,

$$\sigma_k = \sigma_{k-1} + hD(v_k, \sigma_k), \tag{9}$$

which is sometimes referred to as the *backward Euler method*, while when m is one,

$$\sigma_{k+1} = \sigma_k + h\frac{1}{2}\left[D(v_{k+1}, \sigma_{k+1}) + D(v_k, \sigma_k)\right], \tag{10}$$

which is sometimes referred to as the *trapezoidal rule*. Note that these equations only define the solutions implicitly; that is, equations (9) and (10) must be solved numerically for σ_k and σ_{k+1}, respectively.

BDFs constitute the main other way to solve ODEs. BDFs are linear multi-step methods which are especially useful when solving stiff differential equations. From above, we know that, given equation (7), for step size h, a linear multi-step method can, in general, be written as

$$\begin{aligned}\sigma_{k+m} &+ \lambda_{m-1}\sigma_{k+m-1} + \lambda_{m-2}\sigma_{k+m-2} + \cdots + \lambda_0\sigma_k \\ &= h\left[\kappa_m D(v_{k+m}, \sigma_{k+m}) + \kappa_{m-1} D(v_{k+m-1}, \sigma_{k+m-1}) + \cdots + \kappa_0 D(v_k, \sigma_k)\right].\end{aligned}$$

BDFs involve setting κ_i to zero for any i other than m, so a general BDF is

$$\sigma_{k+m} + \lambda_{m-1}\sigma_{k+m-1} + \lambda_{m-2}\sigma_{k+m-2} + \cdots + \lambda_0\sigma_k = h\kappa_m D_{k+m}$$

where D_{k+m} denotes $D(v_{k+m}, \sigma_{k+m})$. Note that, like Adams–Moulten methods, BDFs are implicit methods as well: one must solve nonlinear equations at each step—again, using fixed-point iteration or variants of the Newton–Raphson method. Thus, the methods can be computationally burdensome. However, the evaluation of σ at v_{k+m} in $D(\cdot)$ is an effective way in which to discipline approximate solutions to stiff differential equations.

With this vocabulary in mind, and having outlined how numerical methods can be applied in the simplest model of an auction, we are now in a position to complicate matters considerably. We proceed in stages. We next consider the case where bidders are *ex ante* asymmetric. In such an environment, a number of complications arise. In particular, unlike the model with identical bidders presented above, typically no closed-form expression for the bidding strategies exists in an asymmetric environment (except in a few special cases described below), so numerical methods are required.

2.4. Bidders from Different Urns

Consider a first-price auction with just two potential buyers in the absence of a reserve price and assuming risk neutrality. We present the two-bidder case first because it allows us to highlight the interdependence among bidders and characterize explicitly many features of the first-price auction within the IPVP when bidders are asymmetric. In particular, we contrast features of this problem with those of the symmetric case presented in the previous subsection. Suppose that bidder 1 gets an independent draw from urn 1, denoted $F_1(v_1)$, while bidder 2 gets an independent draw from urn 2, denoted $F_2(v_2)$. Assume that the two valuation distributions have the same support $[\underline{v}, \overline{v}]$. The largest of the two bids wins the auction, and the winner pays what he bid.

Now, $U_1(s_1)$, the expected profit of bid s_1 to player 1, can be written as

$$U_1(s_1) = (v_1 - s_1)\Pr(\text{win}|s_1),$$

while $U_2(s_2)$, the expected profit of bid s_2 to player 2, can be written as

$$U_2(s_2) = (v_2 - s_2)\Pr(\text{win}|s_2).$$

Assuming each potential buyer n is using a bid s_n equal to $\sigma_n(v_n)$ that is monotonically increasing in his value v_n, we can write the probability of winning the auction as

$$\begin{aligned}\Pr(\text{win}|s_n) &= \Pr(S_m < s_n)\\ &= \Pr[\sigma_m(V_m) < s_n]\\ &= \Pr[V_m < \sigma_m^{-1}(s_n)]\\ &= \Pr[V_m < \varphi_m(s_n)]\\ &= F_m[\varphi_m(s_n)].\end{aligned}$$

Thus, the expected profit function for bidder 1 is

$$U_1(s_1) = (v_1 - s_1)F_2[\varphi_2(s_1)],$$

while the expected profit function for bidder 2 is

$$U_2(s_2) = (v_2 - s_2)F_1[\varphi_1(s_2)].$$

The presence of bidder m's inverse-bid function in bidder n's objective makes clear the trade-off bidder n faces: by submitting a lower bid, he increases the profit he receives when he wins the auction, but he decreases his probability of winning the auction.

To construct the pair of Bayes–Nash equilibrium-bid functions, first maximize each expected profit function with respect to its argument. The necessary, first-order condition for these maximization problems are:

$$\begin{aligned}\frac{\mathrm{d}U_1(s_1)}{\mathrm{d}s_1} &= -F_2[\varphi_2(s_1)] + (v_1 - s_1)f_2[\varphi_2(s_1)]\frac{\mathrm{d}\varphi_2(s_1)}{\mathrm{d}s_1} = 0\\ \frac{\mathrm{d}U_2(s_2)}{\mathrm{d}s_2} &= -F_1[\varphi_1(s_2)] + (v_2 - s_2)f_1[\varphi_1(s_2)]\frac{\mathrm{d}\varphi_1(s_2)}{\mathrm{d}s_2} = 0.\end{aligned}$$

Now, a Bayes–Nash equilibrium is characterized by the following pair of differential equations:

$$\begin{aligned}\frac{\mathrm{d}\varphi_2(s_1)}{\mathrm{d}s_1} &= \frac{F_2[\varphi_2(s_1)]}{[\varphi_1(s_1) - s_1]f_2[\varphi_2(s_1)]}\\ \frac{\mathrm{d}\varphi_1(s_2)}{\mathrm{d}s_2} &= \frac{F_1[\varphi_1(s_2)]}{[\varphi_2(s_2) - s_2]f_1[\varphi_1(s_2)]}.\end{aligned} \tag{11}$$

These differential equations allow us to describe some essential features of the problem. First, as within the symmetric IPVP, each individual equation constitutes a first-order differential equation as the highest derivative term in each equation is the first derivative of the function of interest. Unlike within the symmetric IPVP, however, the functions we seek are the inverse-bid functions $\varphi_n(\cdot)$, not the bid functions $\sigma_n(\cdot)$ themselves. While we would like to solve for the bid functions, it is typically impossible within the asymmetric IPVP. In both cases, the first-order conditions that obtain from bidders maximizing their expected profit involve the inverse-bid function and its derivative. However, within the symmetric IPVP, we were concerned with an equilibrium in which all (homogenous) bidders adopted the same bidding strategy $\sigma(v)$. This, together with monotonicity of the bid function, allowed us to map the first-order condition from a differential equation characterizing the inverse-bid function $\varphi(s)$ to a differential equation characterizing the bid function $\sigma(v)$.[6] In the asymmetric environment, this is typically impossible to do because, in general,

$$\varphi_1(s) \neq \varphi_2(s).$$

The inverse-bid functions $\varphi_m(s)$ are helpful because they allow us to express the probability of winning the auction for any choice s: bidder n considers the probability that the other bidder will draw a valuation that will induce him to submit a lower bid in equilibrium than the bid player n submits. Because the bidders draw valuations from different urns, they do not use the same bidding strategy: the valuation which is optimal to submit a bid s is, in general, different for the two bidders. Furthermore, because we can no longer translate either differential equation into one that characterizes the bid function directly, neither differential equation is linear. Finally, note that each differential equation involves a bid s, the derivative of the inverse-bid function for one of the players, which we shall denote hereafter by $\varphi_n'(s)$, and the inverse-bid functions of each of the bidders $\varphi_1(s)$ as well as $\varphi_2(s)$: mathematicians would refer to this system of ODEs as *nonautonomous* because the system involves the bid s explicitly.[7] This last fact highlights the interdependence among players that is common to game-theoretic models. Thus, in terms of deriving the equilibrium inverse-bid functions within the asymmetric IPVP, we must solve a nonlinear system of first-order ODEs.

The case in which each of two bidders draws his valuation from a different urn has allowed us to contrast the features of the problem with those of the symmetric environment in a transparent way. There are also conditions that the equilibrium bid functions must satisfy, and which allow us to solve the pair of differential equations, but we delay that discussion until after we present the N-bidder case.

2.5. General Model

We now extend the model of the first-price auction presented above to one with N potential buyers in the absence of a reserve price and assuming risk neutrality. Suppose that bidder n gets an independent draw from urn n, denoted $F_n(v_n)$. Assume that all valuation distributions have a common, compact support $[\underline{v}, \overline{v}]$. The largest of the N bids wins the auction, and the bidder pays what he bid.

Again, $U_n(s_n)$, the expected profit of bid s_n to player n, can be written as

$$U_n(s_n) = (v_n - s_n)\Pr(\text{win}|s_n).$$

[6] Essentially, this was just an application of the implicit function theorem.

[7] A nonautonomous ODE is also referred to as *time*-dependent, although, for our purposes, *bid*-dependent is a better characterization.

Assuming each potential buyer n is using a bid $\sigma_n(v_n)$ that is monotonically increasing in his value v_n, we can write the probability of winning the auction as

$$\begin{aligned}
\text{Pr(win}|s_n) &= \Pr(S_1 < s_n, S_2 < s_n, \ldots, S_{n-1} < s_n, S_{n+1} < s_n, \ldots, S_N < s_n) \\
&= \Pr\left[(S_1 < s_n) \cap (S_2 < s_n) \cap \ldots \cap (S_{n-1} < s_n) \cap (S_{n+1} < s_n) \cap \ldots \cap (S_N < s_n)\right] \\
&= \prod_{m \neq n} \Pr(S_m < s_n) \\
&= \prod_{m \neq n} \Pr\left[\sigma_m(V_m) < s_n\right] \\
&= \prod_{m \neq n} \Pr\left[V_m < \sigma_m^{-1}(s_n)\right] \\
&= \prod_{m \neq n} F_m\left[\sigma_m^{-1}(s_n)\right] \\
&= \prod_{m \neq n} F_m\left[\varphi_m(s_n)\right].
\end{aligned}$$

Thus, the expected profit function for bidder n is

$$U_n(s_n) = (v_n - s_n) \prod_{m \neq n} F_m[\varphi_m(s_n)].$$

To construct the Bayes–Nash equilibrium-bid functions, first maximize each expected profit function with respect to its argument. The necessary first-order condition for a representative maximization problem is:

$$\begin{aligned}
\frac{\mathrm{d}U_n(s_n)}{\mathrm{d}s_n} = -\prod_{m \neq n} F_m\left[\varphi_m(s_n)\right] + \\
(v_n - s_n) \sum_{m \neq n} f_m[\varphi_m(s_n)] \frac{\mathrm{d}\varphi_m(s_n)}{\mathrm{d}s_n} \prod_{\ell \neq m} F_\ell[\varphi_\ell(s_n)] = 0.
\end{aligned}$$

Replacing s_n with a generic bid s and noting that $\varphi_m(s)$ equals v, we can rearrange this first-order condition as

$$\frac{1}{\varphi_n(s) - s} = \sum_{m \neq n} \frac{f_m[\varphi_m(s)]}{F_m[\varphi_m(s)]} \varphi_m'(s), \tag{12}$$

which can be summed over all N bidders to yield

$$\sum_{m=1}^{N} \frac{1}{\varphi_m(s) - s} = (N-1) \sum_{m=1}^{N} \frac{f_m[\varphi_m(s)]}{F_m[\varphi_m(s)]} \varphi_m'(s)$$

or

$$\frac{1}{(N-1)} \sum_{m=1}^{N} \frac{1}{\varphi_m(s) - s} = \sum_{m=1}^{N} \frac{f_m[\varphi_m(s)]}{F_m[\varphi_m(s)]} \varphi_m'(s).$$

Subtracting equation (12) from this latter expression yields

$$\left[\frac{1}{(N-1)} \sum_{m=1}^{N} \frac{1}{\varphi_m(s) - s}\right] - \frac{1}{\varphi_n(s) - s} = \frac{f_n[\varphi_n(s)]}{F_n[\varphi_n(s)]} \varphi_n'(s),$$

which leads to the, perhaps traditional, differential equation formulation

$$\varphi_n'(s) = \frac{F_n[\varphi_n(s)]}{f_n[\varphi_n(s)]}\left\{\left[\frac{1}{(N-1)}\sum_{m=1}^{N}\frac{1}{\varphi_m(s)-s}\right]-\frac{1}{\varphi_n(s)-s}\right\} \quad n = 1,2,\ldots,N. \tag{13}$$

In addition to this system of differential equations (or system (11) in the two-bidder case presented earlier), two types of boundary conditions exist. The first generalizes the initial condition from the symmetric environment to an asymmetric one:

Left-Boundary Condition on Bid Functions: $\sigma_n(\underline{v}) = \underline{v}$ for all $n = 1, 2, \ldots, N$.

This left-boundary condition simply requires any bidder who draws the lowest valuation possible to bid his valuation. It extends the condition from the environment where there was only one type of bidder to one where there are N types of bidders.[8] We shall need to use the boundary condition(s) with our system of differential equations to solve for the inverse-bid functions, as discussed above. Specifically, we shall be interested in solving for a monotone pure-strategy equilibrium (MPSE) in which each bidder adopts a bidding strategy that maximizes expected payoffs given the strategies of the other players. Given this focus, we can translate the left-boundary condition defined above into the following boundary condition which involves the inverse-bid functions:

Left-Boundary Condition on Inverse-Bid Functions: $\varphi_n(\underline{v}) = \underline{v}$ for all $n = 1, 2, \ldots, N$.

The second type of condition obtains at the right-boundary. Specifically,

Right-Boundary Condition on Bid Functions: $\sigma_n(\overline{v}) = \overline{s}$ for all $n = 1, 2, \ldots, N$.

The reader may find this condition somewhat surprising: even though the bidders may adopt different bidding strategies, all bidders will choose to submit the same bid when they draw the highest valuation possible. No bidder will submit a bid that exceeds the highest bid chosen by all other players because the bidder could strictly decrease the bid by some small amount ε and still win the auction with certainty, and increase his expected profits. This right-boundary condition also has a counterpart which involves the inverse-bid functions

Right-Boundary Condition on Inverse-Bid Functions: $\varphi_n(\overline{s}) = \overline{v}$ for all $n = 1, 2, \ldots, N$.

A few comments are in order here: first, because we now have conditions at both the low and high valuations (bids), our problem is no longer an initial-value problem, but rather a boundary-value problem. Thus, we are interested in a solution to the system of differential equations which satisfies both the left-boundary condition on the inverse-bid function and the right-boundary condition on the inverse-bid function. In the mathematics literature, this is known specifically as a *two-point boundary-value* problem. The critical difference between an initial-value problem and a boundary-value problem is that auxiliary conditions concern the solution at one point in an initial-value problem, while auxiliary conditions concern the solution at several points (in our case, two) in a boundary-value problem. The other challenging component of this problem is that the common high bid $\overline{s}$ is unknown *a priori*, and is determined endogenously by the behavior of bidders. This means that the high bid $\overline{s}$ must be solved for as part of the solution to the system

[8] If a reserve price r_0 existed, then $\sigma_n(r_0) = r_0$ would be the relevant condition: the marginal bidder would bid the reserve price r_0.

of differential equations. That is, we have a system of differential equations that are defined over a domain (because we are solving for the inverse-bid functions) that is unknown *a priori*. In this sense, our problem is considered a *free boundary-value* problem. Note, too, that this system is overidentified: while there are N differential equations, there are $2N$ boundary conditions as well. In addition, some properties of the solution are known beforehand: bidders should not submit bids that exceed their valuations and the (inverse-) bid functions must be monotonic. Of course, the latter implies the former given $\bar{s} \leq \bar{v}$.

One feature of this system of differential equations that makes them interesting to computational economists, and challenging to economic theorists, is that the Lipschitz condition does not hold. A function $g : \mathbb{R}^d \to \mathbb{R}^d$ satisfies the Lipschitz condition on a d-dimensional interval I if there exists a Lipschitz constant τ (greater than zero) such that

$$||g(\mathbf{y}) - g(\mathbf{x})|| \leq \tau||\mathbf{y} - \mathbf{x}||$$

for a given vector norm $\|\cdot\|$ and for all $\mathbf{x} \in I$ and $\mathbf{y} \in I$. To get a better understanding of this, assume $g : \mathbb{R} \to \mathbb{R}$ and rewrite the Lipschitz condition as

$$\left|\frac{g(x+h) - g(x)}{h}\right| \leq \tau$$

where y equals $(x + h)$ and we have chosen to use the L_1 norm.[9] If we assume that $g(\cdot)$ is differentiable and we let $h \to 0$, then the Lipschitz condition means that

$$|g'(x)| \leq \tau,$$

so the derivative is bounded by the Lipschitz constant.[10] The system (13) does not satisfy the Lipschitz condition in a neighborhood of $\underline{v}$ because a singularity obtains at $\underline{v}$. To see this, note that the left-boundary condition requires that $\varphi_n(\underline{v})$ equals $\underline{v}$ for all bidders n equal to $1, \ldots, N$. This condition implies that the denominator terms in the right-hand side of these equations which involve $[\varphi_n(s) - s]$ vanish. Note, too, that the numerators contain $F_n(\cdot)$s, which equal zero at $\underline{v}$. Because the Lipschitz condition is not satisfied for the system, much of the theory concerning systems of ODEs no longer applies.

While not the focus of this research, our presentation would be incomplete were we to ignore the results concerning existence and uniqueness of equilibria developed by economic theorists. The issue of existence is critical to resolve *before* solution methods can be applied. While computational methods could be used to approximate numerically a solution that may not exist, the value of a numerical solution is far greater when we know a solution exists than when not. The issue of uniqueness is essential to empirical researchers using data from first-price auctions. Without uniqueness, an econometrician would have a difficult task justifying that the data observed are all derived from the same equilibrium. Because the Lipschitz condition fails, one of the sufficient conditions of the Picard–Lindelöf theorem, which guarantees that a unique solution exists to an initial-value problem, does not hold. Consequently, fundamental theorems for systems of differential equations cannot be applied to the system (13). Despite this difficulty,

[9] All norms are equivalent in finite-dimensional spaces, so if a function satisfies the Lipschitz condition in one norm, it satisfies the Lipschitz condition in all norms. The Lipschitz constant τ, however, does depend on the choice of norm.

[10] While, in this example, we have assumed that $g(\cdot)$ is differentiable, we have done so for illustrative purposes only. We do not claim that the inverse-bid functions are differentiable everywhere, something we discuss below.

Lebrun [36] proved that the inverse-bid functions are differentiable on $(\underline{v}, \overline{s}]$ and that a unique Bayes–Nash equilibrium exists when all valuation distributions have a mass point at $\underline{v}$ and the value distributions have a common support (as we have assumed above). Existence was also demonstrated by Maskin and Riley [44], while Maskin and Riley [43] investigated some equilibrium properties of asymmetric first-price auctions. The discussion above is most closely related to the approach taken by Lebrun [36], as well as Lizzeri and Persico [38], for auctions with two asymmetric bidders; these researchers established existence by showing that a solution exists to the system of differential equations.[11] Reny [51] proved existence in a general class of games and Athey [1] proved that a pure strategy Nash equilibrium exists for first-price auctions with heterogenous bidders under a variety of circumstances, some of which we consider below. For a discussion on the existence of equilibrium in first-price auctions, see Appendix G of Krishna [33].

2.6. Special Case

As we noted above, an explicit solution to the system (13), or system (11) in the two-bidder case, exists only in a few special cases. The special case we present here involves two bidders who draw valuations from asymmetric uniform distributions.[12] For two uniform distributions to be different, they must have different supports, which requires us to modify slightly the model we presented above. Specifically, consider a first-price auction involving two risk-neutral bidders at which no reserve price exists. Suppose that bidder n gets an independent draw from a uniform distribution $F_n(v_n)$ having support $[\underline{v}, \overline{v}_n]$. For convenience, we assume the lowest possible valuation $\underline{v}$ is zero, and is common to all bidders: the bidders only differ by the highest possible valuation they can draw. The largest of the two bids wins the auction, and the bidder pays what he bid.

Within this environment,

$$F_n(v) = \frac{v}{\overline{v}_n} \quad n = 1, 2,$$

so the probability of bidder n winning the auction with a bid s_n equals

$$\Pr(\text{win}|s_n) = \frac{\varphi_m(s_n)}{\overline{v}_m} \quad m \neq n.$$

Thus, the expected profit function for bidder 1 is

$$U_1(s_1) = (v_1 - s_1)\frac{\varphi_2(s_1)}{\overline{v}_2},$$

while the expected profit function for bidder 2 is

$$U_2(s_2) = (v_2 - s_2)\frac{\varphi_1(s_2)}{\overline{v}_1}.$$

[11] In contrast, Lebrun [35], Maskin and Riley [44] as well as Athey [1] proved the existence of a MPSE under the restriction that bidders can only bid in discrete amounts. To wit, bids must belong to a finite set. These researchers then used a limiting argument, which involves shrinking the minimum bid increment and showing that a sequence of pure strategy equilibria converges uniformly almost everywhere to a pure strategy in which bids are unrestricted.

[12] Our example follows Krishna [33] as well as Maskin and Riley [43]; these researchers compared revenue and efficiency at an asymmetric first-price auction with that of a second-price auction. The derivation of this example was originally provided by Griesmer et al. [20]. Kaplan and Zamir [29] generalized this work by considering two bidders at auction (with or without a reserve price) who each draw valuations from uniform distributions with any asymmetric, bounded supports. Another example, which involves power distributions, was originally derived by Plum [50]. We postpone a discussion of this case as we shall return to it below when we consider bidder collusion.

Taking the first-order conditions for maximization of each bidder's expected profit and setting them equal to zero yields:

$$\frac{\mathrm{d}U_1(s_1)}{\mathrm{d}s_1} = -\frac{\varphi_2(s_1)}{\overline{v}_2} + (v_1 - s_1)\frac{1}{\overline{v}_2}\frac{\mathrm{d}\varphi_2(s_1)}{\mathrm{d}s_1} = 0$$
$$\frac{\mathrm{d}U_2(s_2)}{\mathrm{d}s_2} = -\frac{\varphi_1(s_2)}{\overline{v}_1} + (v_2 - s_2)\frac{1}{\overline{v}_1}\frac{\mathrm{d}\varphi_1(s_2)}{\mathrm{d}s_2} = 0.$$

The following pair of differential equations characterizes the Bayes–Nash equilibrium:

$$\begin{aligned}\varphi_2'(s_1) &= \frac{\varphi_2(s_1)]}{[\varphi_1(s_1) - s_1]} \\ \varphi_1'(s_2) &= \frac{\varphi_1(s_2)]}{[\varphi_2(s_2) - s_2]}.\end{aligned} \tag{14}$$

As described above, the equilibrium inverse-bid functions solve this pair of differential equations, subject to the following boundary conditions:

$$\varphi_n(0) = 0, \quad n = 1, 2,$$

and

$$\varphi_n(\overline{s}) = \overline{v}_n, \quad n = 1, 2.$$

Together, these conditions imply that, while the domains of the bid functions differ, the domains of the inverse-bid functions are the same for both bidders.

This system can be solved in closed-form. The first step is to find the common and, *a priori* unknown, high bid $\overline{s}$. To do this, following Krishna [33], we can rewrite the equation describing $\varphi_n'(s)$ by subtracting one from both sides and rearranging to obtain

$$[\varphi_n'(s) - 1][\varphi_m(s) - s] = \varphi_n(s) - \varphi_m(s) + s.$$

Adding the two equations yields

$$[\varphi_1'(s) - 1][\varphi_2(s) - s] + [\varphi_2'(s) - 1][\varphi_1(s) - s] = 2s.$$

Note, however, that

$$\frac{\mathrm{d}\{[\varphi_1(s) - s][\varphi_2(s) - s]\}}{\mathrm{d}s} = [\varphi_1'(s) - 1][\varphi_2(s) - s] + [\varphi_2'(s) - 1][\varphi_1(s) - s],$$

so

$$\frac{\mathrm{d}\{[\varphi_1(s) - s][\varphi_2(s) - s]\}}{\mathrm{d}s} = 2s.$$

Integrating both sides yields

$$[\varphi_1(s) - s][\varphi_2(s) - s] = s^2 \tag{15}$$

where we have used the left-boundary condition that $\varphi_n(0)$ equals zero to determine the constant of integration. This equation can be used to solve for $\overline{s}$ using the right-boundary condition

$$(\overline{v}_1 - \overline{s})(\overline{v}_2 - \overline{s}) = \overline{s}^2,$$

so

$$\bar{s} = \frac{\bar{v}_1 \bar{v}_2}{\bar{v}_1 + \bar{v}_2}.$$

Following Krishna [33], we can use a change of variables by setting

$$\varphi_n(s) - s = s\psi_n(s)$$

for which

$$\varphi_n'(s) - 1 = \psi_n(s) + s\psi_n'(s).$$

Note, too, that the change of variables implies

$$\frac{\varphi_n(s)}{s} = \psi_n(s) + 1.$$

The key to solving the system of differential equations is that each differential equation in our system (14) can be expressed as an alternative differential equation which depends only on the inverse-bid function, and its derivative, for a single bidder: it does not involve the other bidder's inverse-bid function. To see this, solve for $[\varphi_2(s) - s]$ using equation (15) and substitute it into the equation defining $\varphi_1'(s)$ in the system (14) to obtain

$$\varphi_1'(s) = \frac{\varphi_1(s)[\varphi_1(s) - s]}{s^2}.$$

Now, using the change of variables proposed above, as well as the relationships obtaining from it, this differential equation can be written as

$$\psi_1(s) + s\psi_1'(s) + 1 = \psi_1(s)[\psi_1(s) + 1]$$

which can be rewritten as

$$\psi_1'(s) = \frac{[\psi_1(s)]^2 - 1}{s}.$$

The solution to this differential equation is

$$\psi_1(s) = \frac{1 - k_1 s^2}{1 + k_1 s^2}$$

where k_1 is the constant of integration. Using the change of variables, the inverse-bid function is

$$\varphi_1(s) = \frac{2s}{1 + k_1 s^2}$$

where

$$k_1 = \frac{1}{\bar{v}_1^2} - \frac{1}{\bar{v}_2^2}$$

is determined by the right-boundary condition that $\varphi_1(\bar{s})$ equals $\bar{v}_1$. Likewise,

$$\varphi_2(s) = \frac{2s}{1 + k_2 s^2}$$

where

$$k_2 = \frac{1}{\bar{v}_2^2} - \frac{1}{\bar{v}_1^2}$$

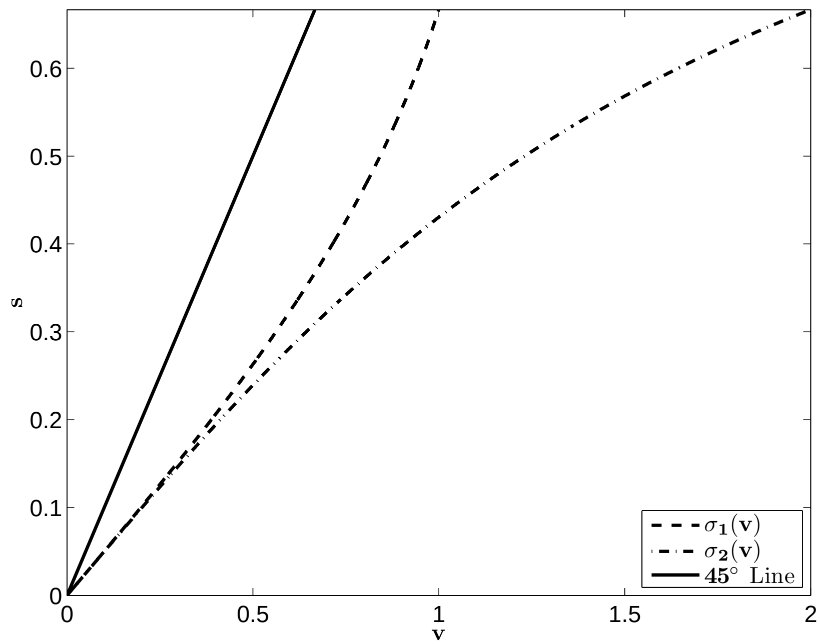

Figure 1: Example Bid Functions for Two Uniform Bidders

which completes our closed-form solution for the inverse-bid functions in this special case. The associated bid functions for the case where $F_1(v)$ is Uniform[0,1] and $F_2(v)$ is Uniform[0,2] are depicted in figure 1.

In this example, tractability obtains because $f_n[\varphi_n(s)]/F_n[\varphi_n(s)]$, the inverse of the Mills' ratio, is a convenient function of the inverse-bid function: it equals $[1/\varphi_n(s)]$. Thus, the pair of differential equations in the two-bidder case can be expressed as a pair of independent ODEs. That is, the relationship among bidders is so special we can use the approach we used to solve for the equilibrium at a symmetric first-price auction. In short, we are able to derive closed-form expressions for the inverse-bid functions (or, likewise, for the bid functions). In general, inverses of the Mills' ratio will involve terms that prevent such isolation and will require using numerical methods.

2.7. Extensions

The model presented above is relevant to a number of different research questions. In this subsection, we discuss some extensions to the model above which also require the use of computational methods. We first cast the model presented above in a procurement environment in which the lowest bidder is awarded the contract. While it may seem reasonable empirically to assume that there may exist more than one type of bidder at auction, the asymmetric first-price model we have presented can arise even when bidders draw valuations from the same distribution. We discuss some situations in which an asymmetric first-price model is relevant, even though bidders draw valuations from the same distribution. In particular, we consider the case of risk-averse bidders, one in which bidders collude and/or form coalitions and, finally, one in which the auctioneer (in our case, the government) grants preference to a class of bidders.[13]

2.7.1. Procurement

We can modify the above analysis of the first-price auction with N potential buyers to a procurement environment in which a government agency seeks to complete an indivisible task at the lowest cost. The agency invites sealed-bid tenders from N potential suppliers—firms. The bids are opened more or less simultaneously and the contract is awarded to the lowest bidder who wins the right to perform the task. The agency then pays the winning firm its bid on completion of the task. Assume that there is no price ceiling—a maximum acceptable bid that has been imposed by the buyer—and assume bidders (firms) are risk neutral. Suppose that bidder n gets an independent cost draw C_n from urn n, denoted $F_n(c_n)$. Assume that all cost distributions have a common, compact support $[\underline{c}, \overline{c}]$.

Now, $U_n(b_n)$, the expected profit of bid b_n to player n, can be written as

$$U_n(b_n) = (b_n - c_n)\,\text{Pr}(\text{win}|b_n).$$

Assuming each potential buyer n is using a bid $\beta_n(c_n)$ that is monotonically increasing in his cost c_n, we can write the probability of winning the auction as

$$\begin{aligned}
\text{Pr}(\text{win}|b_n) &= \text{Pr}(B_1 > b_n, B_2 > b_n, \ldots, B_{n-1} > b_n, B_{n+1} > b_n, \ldots, B_N > b_n) \\
&= \text{Pr}\left[(B_1 > b_n) \cap (B_2 > b_n) \cap \ldots \cap (B_{n-1} > b_n) \cap (B_{n+1} > b_n) \cap \ldots \cap (B_N > b_n)\right] \\
&= \prod_{m \neq n} \text{Pr}\,(B_m > b_n) \\
&= \prod_{m \neq n} \text{Pr}\,[\beta_m(C_m) > b_n] \\
&= \prod_{m \neq n} \text{Pr}\left[C_m > \beta_m^{-1}(b_n)\right] \\
&= \prod_{m \neq n} \left(1 - F_m\left[\beta_m^{-1}(b_n)\right]\right) \\
&\equiv \prod_{m \neq n} \left(1 - F_m\left[\varphi_m(b_n)\right]\right).
\end{aligned}$$

Thus, the expected profit function for bidder n is

$$U_n(b_n) = (b_n - c_n) \prod_{m \neq n} \left(1 - F_m[\varphi_m(b_n)]\right).$$

To construct the Bayes–Nash, equilibrium-bid functions, first maximize each expected profit function with respect to its argument. The necessary, first-order condition for a representative

[13] One can imagine other models which generate the intractable property of asymmetric first-price auctions. For example, if bidders have different budget constraints or, in the case of procurement auctions, different capacity constraints; if bidders must pay an entry fee before they are able to bid and entry fees differ across bidders. Our approach would apply to any type of relaxation of the assumptions or extension to the model we have presented above that will lead bidders to behave differently from one another.

maximization problem is:

$$\frac{\mathrm{d}U_n(b_n)}{\mathrm{d}b_n} = \prod_{m \neq n} \left(1 - F_m\left[\varphi_m(b_n)\right]\right) - \\ (b_n - c_n) \sum_{m \neq n} f_m[\varphi_m(b_n)] \frac{\mathrm{d}\varphi_m(b_n)}{\mathrm{d}b_n} \prod_{\ell \neq m} (1 - F_\ell[\varphi_\ell(b_n)]) = 0.$$

Replacing b_n with a general bid b and noting that $\varphi_m(b)$ equals c, we can rearrange this first-order condition as

$$\frac{1}{b - \varphi_n(b)} = \sum_{m \neq n} \frac{f_m[\varphi_m(b)]}{1 - F_m[\varphi_m(b)]} \varphi_m'(b) \tag{16}$$

which can be summed over all N bidders to yield

$$\sum_{m=1}^{N} \frac{1}{b - \varphi_m(b)} = (N-1) \sum_{m=1}^{N} \frac{f_m[\varphi_m(b)]}{1 - F_m[\varphi_m(b)]} \varphi_m'(b)$$

or

$$\frac{1}{(N-1)} \sum_{m=1}^{N} \frac{1}{b - \varphi_m(b)} = \sum_{m=1}^{N} \frac{f_m[\varphi_m(b)]}{1 - F_m[\varphi_m(b)]} \varphi_m'(b).$$

Subtracting equation (16) from this latter expression yields

$$\left[\frac{1}{(N-1)} \sum_{m=1}^{N} \frac{1}{b - \varphi_m(b)}\right] - \frac{1}{b - \varphi_n(b)} = \frac{f_n[\varphi_n(b)]}{1 - F_n[\varphi_n(b)]} \varphi_n'(b)$$

which leads to the, perhaps traditional, differential equation formulation

$$\varphi_n'(b) = \frac{1 - F_n[\varphi_n(b)]}{f_n[\varphi_n(b)]} \left\{\left[\frac{1}{(N-1)} \sum_{m=1}^{N} \frac{1}{b - \varphi_m(b)}\right] - \frac{1}{b - \varphi_n(b)}\right\}. \tag{17}$$

In addition to this system of differential equations, as in the asymmetric first-price auction, there are two types of boundary conditions on the equilibrium bid functions at an asymmetric procurement auction.

Right-Boundary Condition on Bid Functions: $\beta_n(\bar{c}) = \bar{c}$ for all $n = 1, 2, \ldots, N$.

This right-boundary condition requires any bidder who draws the highest cost possible to bid his cost.[14] We shall need to use the boundary condition(s) with our system of differential equations to solve for the MPSE inverse-bid functions, as discussed above. Given this focus, we can translate this right-boundary condition into the following boundary condition which involves the inverse-bid functions:

Right-Boundary Condition on Inverse-Bid Functions: $\varphi_n(\bar{c}) = \bar{c}$ for all $n = 1, 2, \ldots, N$.

The second type of condition obtains at the left-boundary and is analogous to the right-boundary conditions from the asymmetric first-price auction. Specifically,

[14] If a price ceiling p_0 existed, then $\beta_n(p_0) = p_0$ would be the relevant condition: the marginal bidder would bid the price ceiling p_0.

Left-Boundary Condition on Bid Functions: $\beta_n(\underline{c}) = \underline{b}$ for all $n = 1, 2, \ldots, N$.

This condition requires that, even though the bidders may adopt different bidding strategies, all bidders will choose to submit the same bid if they draw the lowest cost possible. Any bid below $\underline{b}$ would be suboptimal because the firm could strictly increase the bid by some small amount ε and still win the auction with certainty, while at the same time increasing its profits. This left-boundary condition also has a counterpart which involves the inverse-bid functions

Left-Boundary Condition on Inverse-Bid Functions: $\varphi_n(\underline{b}) = \underline{c}$ for all $n = 1, 2, \ldots, N$.

Thus, we are interested in a solution to the system of differential equations which satisfies both the right-boundary condition on the inverse-bid functions and the left-boundary condition on the inverse-bid functions. Because we have conditions on the inverse-bid functions at both ends of the domain, we have a two-point boundary-value problem. In the procurement environment, because the common low bid is unknown *a priori*, the lower boundary constitutes the free boundary.

Again, the system (17) does not satisfy the Lipschitz condition in a neighborhood of $\bar{c}$ because a singularity obtains at $\bar{c}$. To see this, note that right-boundary condition requires that $\varphi_n(\bar{c})$ equals $\bar{c}$ for all bidders n equal to $1, \ldots, N$. This condition implies that the denominator terms in the right-hand side of these equations which involve $[b - \varphi_n(b)]$ vanish. Likewise, the numerators involve a survivor function which equals zero at $\bar{c}$. Thus, again, because the Lipschitz condition is not satisfied, much of the theory concerning systems of ODEs no longer applies.

2.7.2. Risk Aversion

In the discussion of asymmetric first-price auctions that we presented above, we assumed (as researchers most commonly do) that the asymmetry was relevant because bidders drew valuations from different distributions. Alternatively, we could assume that bidders are symmetric in that they all draw valuations from the same distribution, but asymmetric in that they have heterogeneous preferences. Assume that buyer n's value V_n is an independent draw from the (common) cumulative distribution function $F_0(v)$, which is continuous, having an associated positive probability density function $f_0(v)$ that has compact support $[\underline{v}, \bar{v}]$ where $\underline{v}$ is weakly greater than zero. Assume that the number of potential buyers N as well as the cumulative distribution function of values $F_0(v)$ and the support $[\underline{v}, \bar{v}]$ are common knowledge.

We relax the assumption that bidders are risk neutral and, instead, assume that the bidders have different degrees of risk aversion. While individual valuations are private information, all bidders know that valuations are drawn from $F_0(v)$ and know each bidder's utility function. Consider the case in which bidders have constant relative risk aversion (CRRA) utility functions but differ in their Arrow–Pratt coefficient of relative risk aversion

$$(1 - \gamma_n) = \frac{-zW_n''(z)}{W_n'(z)}$$

where $\gamma_n \in (0, 1]$ for all bidders $n = 1, \ldots, N$.[15] Thus, when buyer n submits bid s_n, he receives

[15] While we restrict attention to asymmetric bidders with CRRA utility, Krishna [33] presented the case with symmetric risk-averse bidders having arbitrary utility functions. Our presentation of this model does not mirror those of others, but similar models have been investigated by Cox et al. [12], Matthews [45], Maskin and Riley [42], Matthews [46] as well as Chen and Plott [9].

the following payoff:

$$W_n(s_1, \dots, s_N, v_n) = \begin{cases} (v_n - s_n)^{\gamma_n} & \text{if } s_n > s_m \text{ for all } n \neq m \\ 0 & \text{otherwise.} \end{cases} \tag{18}$$

Under risk neutrality the profit bidder n receives when he wins the auction is linear in his bid s_n so the payoff is additively separable from the bidder's valuation. This breaks down under risk aversion as utility becomes nonlinear—utility is concave in our CRRA case.[16]

Assuming each potential buyer n is using a bid $\sigma_n(v_n)$ that is monotonically increasing in his value v_n, the expected utility function for bidder n is

$$U_n(s_n) = (v_n - s_n)^{\gamma_n} \prod_{m \neq n} F_0[\varphi_m(s_n)].$$

The necessary, first-order condition for a representative utility maximization problem is:

$$\frac{\mathrm{d}U_n(s_n)}{\mathrm{d}s_n} = -\gamma_n (v_n - s_n)^{\gamma_n - 1} \prod_{m \neq n} F_0\left[\varphi_m(s_n)\right] + (v_n - s_n)^{\gamma_n} \sum_{m \neq n} f_0[\varphi_m(s_n)] \frac{\mathrm{d}\varphi_m(s_n)}{\mathrm{d}s_n} \prod_{\ell \neq m} F_0[\varphi_\ell(s_n)] = 0.$$

Replacing s_n with a general bid s and noting that $\varphi_m(s)$ equals v, we can rearrange this first-order condition as

$$\frac{\gamma_n}{\varphi_n(s) - s} = \sum_{m \neq n} \frac{f_0[\varphi_m(s)]}{F_0[\varphi_m(s)]} \varphi_m'(s) \tag{19}$$

which can be summed over all N bidders to yield

$$\sum_{m=1}^{N} \frac{\gamma_m}{\varphi_m(s) - s} = (N-1) \sum_{m=1}^{N} \frac{f_0[\varphi_m(s)]}{F_0[\varphi_m(s)]} \varphi_m'(s)$$

or

$$\frac{1}{(N-1)} \sum_{m=1}^{N} \frac{\gamma_m}{\varphi_m(s) - s} = \sum_{m=1}^{N} \frac{f_0[\varphi_m(s)]}{F_0[\varphi_m(s)]} \varphi_m'(s).$$

Subtracting equation (19) from this latter expression yields

$$\left[\frac{1}{(N-1)} \sum_{m=1}^{N} \frac{\gamma_m}{\varphi_m(s) - s}\right] - \frac{\gamma_n}{\varphi_n(s) - s} = \frac{f_0[\varphi_n(s)]}{F_0[\varphi_n(s)]} \varphi_n'(s)$$

which leads to the following differential equation formulation:

$$\varphi_n'(s) = \frac{F_0[\varphi_n(s)]}{f_0[\varphi_n(s)]} \left\{ \left[\frac{1}{(N-1)} \sum_{m=1}^{N} \frac{\gamma_m}{\varphi_m(s) - s}\right] - \frac{\gamma_n}{\varphi_n(s) - s} \right\}. \tag{20}$$

[16] In the limit case, where γ_n equals one for all bidders, this model simplifies to the symmetric IPVP model with risk-neutral bidders.

In addition to this system of ODEs, there are two types of boundary conditions on the equilibrium (inverse-) bid functions which mirror those of the asymmetric first price auction:

Right-Boundary Condition (on Inverse-Bid Functions): $\varphi_n(\overline{s}) = \overline{v}$ for all $n = 1, 2, \ldots, N$

and

Left-Boundary Condition (on Inverse-Bid Functions): $\varphi_n(\underline{v}) = \underline{v}$ for all $n = 1, 2, \ldots, N$.

We are interested in a solution to the system (20) which satisfies the right- and left-boundary conditions on the inverse-bid functions. In general, no closed-form solution exists and the Lipschitz condition does not hold in a neighborhood around $\underline{v}$ because of a singularity. Consequently, numerical methods are again required.

2.7.3. Collusion or Presence of Coalitions

Consider instead a model in which all N potential bidders have homogenous, risk-neutral preferences. Furthermore, assume that all bidders draw independent valuations from the same distribution $F_0(v)$, having an associated positive probability density function $f_0(v)$ that has compact support $[\underline{v}, \overline{v}]$. However, suppose subsets of bidders join (collude to form) coalitions. Introducing collusion into an otherwise symmetric auction is what motivated the pioneering research of Marshall et al. [41]. Bajari [3] proposed using numerical methods to understand better collusive behavior in a series of comparative static-like computational experiments. We discuss the contribution of these researchers later in this chapter. First, however, it is important to recognize that the symmetric first-price auction with collusion, as is typically modelled, is equivalent to the standard asymmetric first-price auction model presented above. Specifically, if the bidders form coalitions of different sizes, a distributional asymmetry is created and the model is just like the case in which each coalition is considered a bidder which draws its valuation from a different distribution.

The N potential bidders form K coalitions with a representative coalition k having size n_k with

$$n_k \geq 1, \text{ for } k = 1, \ldots, K$$

and

$$\sum_{k=1}^{K} n_k = N$$

where K is less than or equal to N. We are not concerned with how the coalition divides up the profit if it wins the item at auction. Instead, we are simply concerned with how each coalition behaves in this case. Note, too, that we allow for coalitions to be of size one; that is, a bidder may choose not to belong to a coalition, and thus behaves independently (noncooperatively).

Assume that each coalition k chooses its bid s_k to maximize its (aggregate) expected profit

$$U_k(s_k) = (v_k - s_k) \Pr(\mathsf{win}|s_k).$$

Coalition k will win the auction with tender s_k when all other coalitions bid less than s_k because the highest valuation of the object for each rival coalition induces bids that are less than that of coalition k. Assuming each coalition k adopts a bidding strategy $\sigma_k(v_k)$ that is monotonically increasing in its value v_k, we can write the probability of winning the auction as

$$\begin{aligned}
\Pr(\text{win}|s_k) &= \Pr(S_1 < s_k, S_2 < s_k, \ldots, S_{k-1} < s_k, S_{k+1} < s_k, \ldots, S_K < s_k) \\
&= \Pr\left[(S_1 < s_k) \cap (S_2 < s_k) \cap \ldots \cap (S_{k-1} < s_k) \cap (S_{k+1} < s_k) \cap \ldots \cap (S_K < s_k)\right] \\
&= \prod_{j \neq k} \Pr\left(S_j < s_k\right) \\
&= \prod_{j \neq k} \Pr\left[\sigma_j(V_j) < s_k\right] \\
&= \prod_{j \neq k} \Pr\left[V_j < \sigma_j^{-1}(s_k)\right] \\
&= \prod_{j \neq k} F_0\left[\sigma_j^{-1}(s_k)\right]^{n_j} \\
&= \prod_{j \neq k} F_0\left[\varphi_j(s_k)\right]^{n_j}
\end{aligned}$$

where, again, $\varphi(\cdot)$ is the inverse-bid function. Thus, the expected profit function of coalition k is

$$U_k(s_k) = (v_k - s_k) \prod_{j \neq k} F_0[\varphi_j(s_k)]^{n_j}.$$

When the number of bidders in each coalition is different for at least two coalitions (when $n_j \neq n_k$ for some $j \neq k$), then even though all bidders draw valuations from the same distribution, a distributional asymmetry obtains. Thus, for a given bid, each coalition faces a different probability of winning the auction. This probability of winning differs across coalitions because, when choosing its bid, each coalition k must consider the distribution of the *maximum* of n_j draws for each rival coalition $j \neq k$. If all coalitions are of the same size, then this model collapses to the symmetric IPVP with K bidders for which we can solve for the (common) bidding strategy which has a closed-form solution, as shown above. However, when the number of bidders in each coalition is different for at least two of the coalitions, the model is just like the asymmetric first-price model.

Each coalition will choose its bid, given its (highest) valuation, to maximize its expected profit. The necessary, first-order condition for a representative maximization problem is:

$$\begin{aligned}
\frac{\mathrm{d}U_k(s_k)}{\mathrm{d}s_k} = -\prod_{j \neq k} F_0\left[\varphi_j(s_k)\right]^{n_j} + \\
(v_k - s_k) \sum_{j \neq k} n_j f_0[\varphi_j(s_k)] F_0[\varphi_j(s_k)]^{n_j - 1} \frac{\mathrm{d}\varphi_j(s_k)}{\mathrm{d}s_k} \prod_{\ell \neq j} F_0[\varphi_\ell(s_k)] = 0.
\end{aligned}$$

Replacing s_k with a general bid s and noting that $\varphi_j(s)$ equals v, we can rearrange this first-order condition as

$$\frac{1}{\varphi_k(s) - s} = \sum_{j \neq k} \frac{n_j f_0[\varphi_j(s)]}{F_0[\varphi_j(s)]} \varphi_j'(s) \tag{21}$$

which can be summed over all K coalitions to yield

$$\sum_{j=1}^{K} \frac{1}{\varphi_j(s) - s} = (K - 1) \sum_{j=1}^{K} \frac{n_j f_0[\varphi_j(s)]}{F_0[\varphi_j(s)]} \varphi_j'(s)$$

or

$$\frac{1}{(K-1)}\sum_{j=1}^{K}\frac{1}{\varphi_j(s)-s} = \sum_{j=1}^{K}\frac{n_j f_0[\varphi_j(s)]}{F_0[\varphi_j(s)]}\varphi_j'(s).$$

Subtracting equation (21) from this latter expression yields

$$\left[\frac{1}{(K-1)}\sum_{j=1}^{K}\frac{1}{\varphi_j(s)-s}\right] - \frac{1}{\varphi_k(s)-s} = \frac{n_k f_0[\varphi_k(s)]}{F_0[\varphi_k(s)]}\varphi_k'(s)$$

which leads to the, perhaps traditional, differential equation formulation

$$\varphi_k'(s) = \frac{F_0[\varphi_k(s)]}{n_k f_0[\varphi_k(s)]}\left\{\left[\frac{1}{(K-1)}\sum_{j=1}^{K}\frac{1}{\varphi_j(s)-s}\right] - \frac{1}{\varphi_k(s)-s}\right\}.$$

In addition to this system of ODEs, there are two types of boundary conditions on the equilibrium (inverse-) bid functions which mirror those of the asymmetric first price auction:

Right-Boundary Condition (on Inverse-Bid Functions): $\varphi_k(\overline{s}) = \overline{v}$ for all $k = 1, 2, \ldots, K$

and

Left-Boundary Condition (on Inverse-Bid Functions): $\varphi_k(\underline{v}) = \underline{v}$ for all $k = 1, 2, \ldots, K$.

In this collusive environment, where bidders form coalitions, there is almost never a closed-form solution to the system of ODEs: one exception is when the bidders all draw valuations from a common uniform distribution. In such an environment, it is as if each coalition k receives a draw from a power distribution with parameter (power) n_k and the coalition game is like an asymmetric first-price auction in which each bidder (coalition) receives a draw from a different power distribution. Plum [50] derived the explicit equilibrium bid functions within such an environment when there are two bidders (or, in our case, coalitions) at auction (in fact, he allowed for the support of the valuations to vary across bidders as well).[17] This uniform/power distribution example constitutes another very special case of an asymmetric auction. In general, no closed-form solution exists and the Lipschitz condition does not hold in a neighborhood around $\underline{v}$ because of a singularity. Again, numerical methods are required.

2.7.4. Bid Preferences

Even when bidders draw valuations or costs from the same distribution, buyers (sellers) sometimes invoke policies or rules that introduce asymmetries. Bid preference policies are a commonly-studied example; see, for example, Marion [40], Hubbard and Paarsch [23], as well as Krasnokutskaya and Seim [32]. We shall continue with our procurement model presented earlier by considering the effect of a bid preference policy. Specifically, consider the most commonly-used preference programme under which the bids of preferred firms are treated differently for the purposes of evaluation only. In particular, the bids of preferred firms are typically scaled by some discount factor which is one plus a preference rate denoted ρ. Suppose there are N_1 preferred

[17] Marshall et al. [41] have also provided a partial characterization of the equilibrium bid functions in such an environment in Appendix A of their paper. See, too, Cheng [11] for such a derivation, which includes a nice discussion of the relationship between the uniform and power distributions as well as revenue comparisons across auction models.

bidders and N_2 typical (nonpreferred) bidders, where $(N_1 + N_2)$ equals N. The preference policy reduces the bids of class 1 firms for the purposes of evaluation only; a winning firm is still paid its bid, on completion of an awarded contract.

Each bidder draws a firm-specific cost independently from a potentially asymmetric cost distribution $F_i(c)$ where i corresponds to the class the firm belongs to $\{1, 2\}$. Each firm then chooses its bid b to maximize its

$$U_i(b_i) = (b_i - c_i)\,\mathrm{Pr}(\mathrm{win}|b_i).$$

Suppose that all bidders of class i use a (class-symmetric) monotonically increasing strategy $\beta_i(\cdot)$. This assumption imposes structure on the probability of winning an auction, conditional on a particular strategy $\beta_i(\cdot)$, which then determines the bid b_i given a class i firm's cost draw. In particular, for a class 1 bidder,

$$\mathrm{Pr}(\mathrm{win}|b_1) = (1 - F_1[\varphi_1(b_1)])^{N_1-1}\left(1 - F_2\left[\varphi_2\left(\frac{b_1}{1+\rho}\right)\right]\right)^{N_2},$$

while for a class 2 bidder

$$\mathrm{Pr}(\mathrm{win}|b_2) = [1 - F_1\left(\varphi_1\left[(1+\rho)b_2\right]\right)]^{N_1}\,(1 - F_2[\varphi_2(b_2)])^{N_2-1}$$

where $\varphi_i(\cdot)$ equals $\beta_i^{-1}(\cdot)$. These probabilities follow the derivations we have presented above after accounting for the fact that preferred (nonpreferred) bidders inflate (discount) tenders from bidders in the rival class in considering the valuation required of opponents from that class to induce a bid that would win the auction. Substituting these probabilities into the expected profit for a firm belonging to class i and taking first-order conditions yields

$$\frac{\mathrm{d}U_1(b_1)}{\mathrm{d}b_1} = 1 - [b_1 - \varphi_1(b_1)]\left[\frac{(N_1-1)\,f_1\left[\varphi_1(b_1)\right]\varphi_1'(b_1)}{1 - F_1\left[\varphi_1(b_1)\right]} + \frac{N_2\,f_2\left[\varphi_2\left(\frac{b_1}{1+\rho}\right)\right]\frac{1}{1+\rho}\varphi_2'\left(\frac{b_1}{1+\rho}\right)}{1 - F_2\left[\varphi_2\left(\frac{b_1}{1+\rho}\right)\right]}\right] = 0$$

and

$$\frac{\mathrm{d}U_2(b_2)}{\mathrm{d}b_2} = 1 - [b_2 - \varphi_2(b_2)]\left[\frac{N_1\,f_1\left(\varphi_1\left[(1+\rho)b_2\right]\right)(1+\rho)\varphi_1'\left[(1+\rho)b_2\right]}{1 - F_1\left(\varphi_1\left[(1+\rho)b_2\right]\right)} + \frac{(N_2-1)\,f_2\left[\varphi_2(b_2)\right]\varphi_2'(b_2)}{1 - F_2\left[\varphi_2(b_2)\right]}\right] = 0$$

Most observed preference policies use a constant preference rate to adjust the bids of qualified firms for the purposes of evaluation only. To incorporate bid preferences in the model, using this common preference rule, the standard boundary conditions must be adjusted to depend on the class of the firm. Reny and Zamir [52] have extended the results concerning existence of equilibrium bid functions in a general asymmetric environment; these results apply to the bid-preference case. Under the most common preference policy, the equilibrium inverse-bid functions will satisfy the class-specific conditions which are revised from the general procurement model presented above. Specifically,

Right-Boundary Conditions (on Inverse-Bid Functions):

- for all nonpreferred bidders of class 2, $\varphi_2(\bar{c}) = \bar{c}$;
- for all preferred bidders of class 1, $\varphi_1(\bar{b}) = \bar{c}$, where $\bar{b} = \bar{c}$ if $N_1 > 1$, but when $N_1 = 1$, then $\bar{b}$ is determined by

$$\bar{b} = \operatorname*{argmax}_b \left[(b - \bar{c}) \left(1 - F_2 \left[\varphi_2 \left(\frac{b}{1+\rho} \right) \right] \right)^{N_2} \right].$$

These right-boundary conditions specify that, with a preference policy, a nonpreferred bidder will bid its cost when it has the highest cost. When just one preferred firm competes with nonpreferred firms, that firm finds it optimal to submit a bid that is greater than the highest cost because the preference rate will reduce the bid and allow the preferred firm to win the auction with some probability. However, when more than one firm receives preference, it is optimal for preferred firms to bid their costs at the right boundary. These arguments are demonstrated in Appendix A of Hubbard and Paarsch [23].

The left-boundary conditions will also be class-specific when the preference rate ρ is positive. Specifically,

Left-Boundary Conditions (on Inverse-Bid Functions): there exists an unknown bid $\underline{b}$ such that

- for all nonpreferred bidders of class 2, $\varphi_2(\underline{b}) = \underline{c}$;
- for all preferred bidders of class 1, $\varphi_1 \left[(1+\rho)\underline{b} \right] = \underline{c}$.

These left-boundary conditions require that, when a nonpreferred firm draws the lowest cost, it tenders the lowest possible bid $\underline{b}$, whereas a preferred firm submits $(1+\rho)\underline{b}$. This condition can be explained by a similar argument to the standard left-boundary condition, taking into account that preferred bids get adjusted using ρ.

Note, too, that to ensure consistency across solutions, Hubbard and Paarsch [23] as well as Krasnokutskaya and Seim [32] assumed that nonpreferred players bid their costs if those costs are in the range $(\bar{c}/(1+\rho), \bar{c})$. Because of the preferential treatment (and assuming more than one bidder receives preferential treatment), nonpreferred players cannot win the auction when they bid higher than $[\bar{c}/(1+\rho)]$. Thus, any bidding strategy will be acceptable in a Bayes–Nash equilibrium, which is why the assumption is needed for consistency.

In the above model, we have allowed the firms to draw costs from different distributions. If the bidders draw costs from symmetric distributions, but are treated asymmetrically, then we still must solve an asymmetric (low-price) auction as the discrimination among classes of bidders induces them to behave in different ways. Note, too, that unlike the canonical asymmetric auctions presented above, where the asymmetry is exogenously fixed (the distributions are set for the bidders), in an environment with bid preferences and symmetric bidders, an asymmetry obtains which is endogenous as the preference rate ρ is typically a choice variable of the procuring agency. Regardless of the reason, there is no closed-form solution as in the symmetric environment with no preference policy. The Lipschitz condition again does not hold in a neighborhood around $\bar{c}$, so numerical methods are required.

3. Primer on Relevant Numerical Strategies

In this section, we describe several numerical strategies that have been used to solve two-point boundary-value problems that are similar to the ones researchers face in models of asymmetric first-price auctions. We use this section not just as a way of introducing the strategies, but so we can refer to them later when discussing what researchers concerned with solving for (inverse-) bid functions at asymmetric first-price auctions have done.

3.1. Shooting Algorithms

A common way to solve boundary-value problems is to treat them like initial-value problems, solving them repeatedly, until the solution satisfies both boundary conditions. This approach often involves algorithms that are referred to as *shooting*.[18] To understand how shooting algorithms work, consider firing an object at a target some distance away. Suppose that you do not hit the target successfully on your first try. Presumably, if hitting the target is important, then you will learn from your first miss, make appropriate adjustments, and fire again. You will continue to repeat this process until you successfully hit your target. The key characteristics which allow you to hit your target, eventually, are that you know how to fire an object (using whatever mechanism is used to send the object at the target) and that you recognize the type of adjustments that need to be made so that your successive shots at the target improve. This story provides an analogy for the procedure used in a shooting algorithm to solve boundary-value problems.

What features might be attractive for solving a system of ODEs which constitute a two-point boundary-value problem like we have in the asymmetric first-price auctions? The first is that an efficient, and accurate, solution method is used to solve the system of differential equations on the relevant interval. The second is that the researcher knows how to make adjustments to a given solution, so that the next iteration improves on the previous. In a two-point boundary-value problem, conditions are imposed on either end of the interval (typically and, more importantly, in our case). The shooting algorithm treats one of the boundaries like an initial value. Given that initial value, there are well known ways to solve a system of differential equations; we then continue our earlier discussion of methods mathematicians commonly use to solve ODEs. After solving the system and arriving at the other boundary, we check to see whether the other (target) condition is satisfied. If not, then we need to understand how to adjust our initial condition, so that when we re-solve the system of equations, we come closer to satisfying our target condition. Note that, if we do not understand how to make the proper adjustment, then we have little hope of converging to a solution. We discuss the shooting algorithm in the first-price auction as well as the low-price (procurement) auction and, then include a discussion of potential solution techniques that can be used within the shooting algorithm to solve the initial-value problem at each iteration.

First, consider solving for the equilibrium inverse-bid functions at an asymmetric first-price auction. Recall the two boundary conditions we have concerning our equilibrium inverse-bid functions that must hold in this case:

$$\varphi_n(\underline{v}) = \underline{v},\ n = 1, \ldots, N,$$

[18] As we discuss in the next section of the paper, using shooting algorithms was first proposed by Marshall et al. [41] for the case of two bidders who draw valuations from asymmetric power distributions, and generalized by Bajari [3] to the N-bidder case for arbitrary distributions.

and

$$\varphi_n(\bar{s}) = \bar{v}, \; n = 1, \ldots, N.$$

Our first decision is to determine which condition should serve as our initial condition and which should serve as a terminal condition. Note the difference between the two conditions—for the left-boundary, we know both the bid as well as the valuation *a priori*, while for the right-boundary we know only the valuation $\bar{v}$, but not the common high bid $\bar{s}$ for which we must solve. Because we do not know the value of $\bar{s}$ *a priori*, the right-boundary makes a poor target: after solving the system, we shall not know whether our solution involves the correct value for the high bid $\bar{s}$ or how to interpret whether the value(s) obtained were too high or too low (as the true value is unknown) in order to make proper adjustments. Ignoring the issue that the Lipschitz condition does not hold for the system at the lower boundary, the left-boundary condition makes for a good target: we know the bid as well as its corresponding valuation for all players. Thus, we want to use the condition that $\varphi_n(\bar{s})$ equals $\bar{v}$ as our "initial" value. As such, using a shooting algorithm at a first-price auction involves using a *backwards* or *reverse* shooting algorithm in which our initial value is actually our "terminal" value.

For a given $\bar{s}$, the proposed solution can fail in one of two ways, both of which involve evaluating the solution at the target condition and recognizing that the proposed solution (shot) did not hit the target. One type of failure is that the value of at least one of the N approximated inverse-bid functions at $\underline{v}$ is a value that is "too far" from the true (known) value which is $\underline{v}$; i.e., $[\hat{\varphi}_n(\underline{v}) - \underline{v}]$ is too large. This failure obtains when the guess for $\bar{s}$ is too low. In this case, the inverse-bid functions are well-behaved in that they are monotonic, but they do not satisfy the target condition. Consequently, the guess for the unknown high bid $\bar{s}$ must be increased; that is how to adjust from the missed shot. The other type of failure involves the solution "blowing up" or diverging. Specifically, the solutions explode toward minus infinity as the bids approach $\underline{v}$. In this case, the guess for the high bid $\bar{s}$ is too high and the candidate solution never reaches the target condition. Under this type of failure, the appropriate modification involves decreasing the guess for the unknown high bid $\bar{s}$. We illustrate these two failures in figure 2, in which we depict a situation in which the candidate solutions involve the true value $\bar{s}^*$, a value in which the high bid is too low $\bar{s}_L$, and a value in which the high bid is too high $\bar{s}_H$.[19] Note that when $\bar{s}$ is too high, the system approaches the 45° line and singularities obtain. To see this, recall system (13) and note that the denominators of each of the terms in brackets involve $[\varphi_n(s) - s]$. As the inverse-bid function approaches the 45° line, players' bids approach their valuations causing the singularities. Thus, to obtain convergence, bids must be kept below their valuations. Consequently, convergence will obtain from the left of the 45° line.[20]

In a model of a first-price auction, the shooting algorithm for a representative iteration i can be summarized as follows:

1. Take a guess for the common high bid $\bar{s}_i \in [\underline{v}, \bar{v}]$.
2. Solve the system of differential equations backwards on the interval $[\underline{v}, \bar{s}_i]$.
3. Use the value that a valid (monotonic) solution takes at $\underline{v}$ to gauge whether to increase or to decrease the guess $\bar{s}_i$. Specifically,

[19] In this example, there are two asymmetric bidders, which is why there are two functions with the same line style for each legend entry, with $[\underline{v}, \bar{v}]$ equal to $[0, 1]$. We depict the intuition in (s, v)-space as the algorithm is used to find the inverse-bid functions.

[20] See Appendix B of Li and Riley [37].

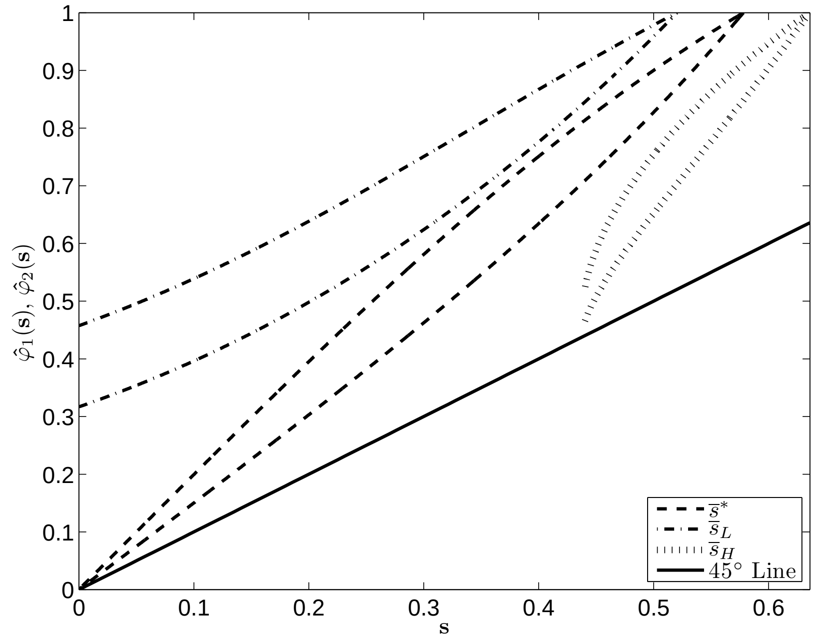

Figure 2: Intuition for (Backwards) Shooting Algorithm at an Asymmetric First-Price Auction

a) if the solution at $\underline{v}$ blows up, then set $\overline{s}_{i+1} < \overline{s}_i$ (decrease $\overline{s}_i$) in step 1 and try again;

b) if the approximated solution at $\underline{v}$ is in $[\underline{v}, \overline{v}]$, but does not meet a pre-specified tolerance criteria for at least one bidder ($[\varphi_n(\underline{v}) - \underline{v}] > \varepsilon$ for some bidder n), then set $\overline{s}_{i+1} > \overline{s}_i$ (increase $\overline{s}$) in step 1 and try again.

4. Stop when

$$\|\hat{\varphi}_n(\underline{v}) - \underline{v}\| \leq \varepsilon \text{ for all } n = 1, \ldots, N$$

for some pre-specified norm $\|\cdot\|$ and pre-specified tolerance level ε.

Of course, the algorithm can be modified and improved. For example, once one iteration has been considered with a high bid that was too high and one has been considered with a high bid that was too low, a bisection routine can be used to speed-up convergence. Of course, bisection is generally considered to have a slow rate of convergence, so a Newton-type iteration or any other root-finding procedure may be preferred.

Solving for the inverse-bid functions at an asymmetric low-price auction is essentially the mirror image of the first-price shooting algorithm. Recall the two boundary conditions we have concerning our inverse-bid functions that must hold in this case:

$$\varphi_n(\underline{c}) = \underline{b} \text{ for } n = 1, \ldots, N,$$

and

$$\varphi_n(\overline{c}) = \overline{c} \text{ for } n = 1, \ldots, N.$$

In this case, the right-boundary makes for a good target as it allows us to evaluate how close the candidate solution is to the true solution. Because, in the procurement environment, the common low bid is unknown *a priori*, we can use this low bid as our initial value in a (forward) shooting algorithm.

The candidate solution can again fail in two ways which correspond to the preceding discussion concerning first-price auctions. One type of failure involves the cost of at least one of the N approximated inverse-bid functions at $\overline{c}$ being too far below the true (known) value which is $\overline{c}$; i.e., $[\overline{c} - \hat{\varphi}_n(\overline{c})]$ is too large. This failure obtains when the guess for $\underline{b}$ is too high. In this case, the inverse-bid functions are well-behaved in that they are monotonic, but they do not satisfy the target condition. Consequently, the guess for the unknown low bid $\underline{b}$ must be decreased. The other type of failure again involves the system diverging, this time toward infinity as the bid approaches $\overline{c}$. In this case, the guess for the low bid $\underline{b}$ is too low and the proposed solution never reaches the target condition. Under this type of failure, the appropriate modification involves increasing the guess for the unknown low bid $\underline{b}$. A formal argument for this procurement setting is provided in Appendix B of Bajari [3]; see Lemmata 7 and 8 of that paper.

The shooting algorithm at a low-price auction for a representative iteration i can be summarized as follows:

1. Take a guess for the common low bid $\underline{b}_i \in [\underline{c}, \overline{c}]$.
2. Solve the system of differential equations on on the interval $[\underline{b}_i, \overline{c}]$.
3. Use the value that a valid (monotonic) solution takes at $\overline{c}$ to gauge whether to increase or decrease the guess $\underline{b}_i$. Specifically,
   a) if the solution at $\overline{c}$ blows up, then set $\underline{b}_{i+1} > \underline{b}_i$ (increase $\underline{b}_i$) in step 1 and try again;
   b) if the approximated solution at $\overline{c}$ is in $[\underline{c}, \overline{c}]$, but does not meet a pre-specified tolerance criteria for at least one bidder ($[\varphi_n(\overline{c}) - \overline{c}] > \varepsilon$ for some bidder n), then set $\underline{b}_{i+1} < \underline{b}_i$ (decrease $\underline{b}$) in step 1 and try again.
4. Stop when
$$||\hat{\varphi}_n(\overline{c}) - \overline{c}|| \leq \varepsilon \text{ for all } n = 1, \ldots, N$$
   for some pre-specified norm $|| \cdot ||$ and pre-specified tolerance level ε.

As we suggested earlier, a root-finding routine can be used to complement this approach, to improve efficiency.

Throughout this subsection, we have taken for granted that the researcher has a viable and stable way of solving the system of ODEs which is subject to initial conditions. Any textbook concerning ODEs (and, most likely, any numerical analysis textbook) will present a number of common ways to approximate the solution to a system of first-order initial-value problems. For a discussion of these methods, with emphasis on applications to problems encountered by economists, see Judd [28], although there is no discussion of auctions in that book. As such, we summarize them only briefly here because the approach taken is often what distinguishes among research concerning asymmetric first-price auctions. Typically, these methods involve approximating the solution to the system of ODEs at a grid of points and then interpolating these values to provide a continuous approximation. This approach means that the system of differential equations are treated like a system of difference equations. The distance between grid points is referred to as the *step size*.

For ease of presentation, let us describe the system of N first-order ODEs for which a representative equation for bidder n was given by (13) as

$$\varphi_1'(s) = g_1\left[s, \varphi_1(s), \varphi_2(s), \ldots, \varphi_N(s)\right]$$
$$\vdots$$
$$\varphi_N'(s) = g_N\left[s, \varphi_1(s), \varphi_2(s), \ldots, \varphi_N(s)\right]$$

which we shall express succinctly as

$$\varphi_1'(s) = g_1\left[s, \boldsymbol{\varphi}(s)\right]$$
$$\vdots$$
$$\varphi_N'(s) = g_N\left[s, \boldsymbol{\varphi}(s)\right]$$

where $\boldsymbol{\varphi}(s)$ collects all of the inverse-bid functions, each evaluated at bid s, and $g_n[s, \boldsymbol{\varphi}(s)]$ represents the right-hand side of the differential equation for bidder n. Consider approximating the solution to this system of ODEs at a grid of bids

$$\underline{v} = s_0 < s_1 < \cdots < s_T = \overline{s}$$

where $(T + 1)$ is the number of points in the grid and

$$s_t = s_0 + th \quad \text{for} \quad t = 0, 1, \ldots, T$$

for step size h. Let s_0 be the bid relevant for the initial condition and let s_T be the bid that is relevant for the target (terminal) condition.[21] Our solution to this system will involve a value V_t^n which approximates the inverse-bid function for player n at bid s_t for each bid in the grid space and for each bidder at auction; i.e., we need to approximate

$$V_t^n = \hat{\varphi}_n(s_t) \text{ for all } t = 1, \ldots, T, \text{ and all } n = 1, \ldots, N.$$

The solution methods we discuss involve first fixing the initial condition to be satisfied for each bidder

$$V_0^n = \varphi_n(s_0) = \varphi_n(\overline{s}) = \overline{v}$$

and, then, approximating the difference equation for $V_1^n, V_2^n, \ldots, V_T^n$, in sequence.

Taylor's method is one of the most intuitively easy ways to understand how to solve such a system. Under the explicit (forward) Taylor's method *of order d*, the approximate value of the inverse-bid function for player n at a bid (step) s_{t+1} can be expressed as

$$V_{t+1}^n = V_t^n + hg_n(s_t, \mathbf{V}_t) + \frac{h^2}{2} g_n'(s_t, \mathbf{V}_t) + \cdots + \frac{h^d}{d!} g_n^{(d-1)}(s_t, \mathbf{V}_t) \tag{22}$$

where $\mathbf{V}_t$ collects $(V_t^1, V_t^2, \ldots, V_t^N)$. This scheme is motivated by a Taylor-series argument. Suppose $\varphi_n(s)$ is the true inverse-bid function for player n. Then, expanding $\varphi_n(s)$ around s_t yields

$$\varphi_n(s_{t+1}) = \varphi_n(s_t) + h\varphi_n'(s_t) + \frac{h^2}{2}\varphi_n''(s_t) + \cdots + \frac{h^d}{d!}\varphi_n^{(d)}(s_t) + \frac{h^{d+1}}{(d+1)!}\varphi_n^{(d+1)}(\xi_t) \tag{23}$$

[21] We cast this discussion in terms of an asymmetric first-price auction in which the initial bid s_0 is $\overline{s}$ and the terminal bid s_T is $\underline{v}$.

for some $\xi_t \in [s_t, s_{t+1}]$. Dropping the h^{d+1} term and assuming $\varphi_n'(s_t)$ equals $g_n(s_t, \mathbf{V}_t)$ as well as V_t^n equals $\varphi_n(s_t)$ yields Taylor's method proposed in equation (22). Note, too, that Euler's method which would approximate V_{t+1}^n by

$$V_{t+1}^n = V_t^n + hg_n(s_t, \mathbf{V}_t)$$

is Taylor's method of order one.

An alternative to the explicit (forward) Taylor's method proposed above is the implicit (backward) Taylor's method (of order d). In the explicit Taylor's method, we used a Taylor-series expansion of $\varphi_n(s)$ around s_t, but we could have considered an expansion around s_{t+1}, instead. Thus,

$$\varphi_n(s_t) \doteq \varphi_n(s_{t+1}) - h\varphi_n'(s_{t+1}) - \frac{h^2}{2}\varphi_n''(s_{t+1}) - \cdots - \frac{h^d}{d!}\varphi_n^{(d)}(s_{t+1})$$

which motivates the implicit Taylor's method

$$V_{t+1}^n = V_t^n + hg_n(s_{t+1}, \mathbf{V}_{t+1}) + \frac{h^2}{2}g_n'(s_{t+1}, \mathbf{V}_{t+1}) + \cdots + \frac{h^d}{d!}g_n^{(d-1)}(s_{t+1}, \mathbf{V}_{t+1}).$$

Thus, V_{t+1}^n is defined only implicitly in terms of s_{t+1} and V_t^n. This scheme requires an n-dimensional system of nonlinear equations to be solved at each step to approximate V_{t+1}^n. While this approach is more expensive in terms of computing time, the approximations are typically much better than those obtained under the explicit Taylor's method. Intuitively, the value V_{t+1}^n depends not only on V_t^n and s_{t+1}, but also on the behavior of $g_n(s, \mathbf{V})$ at $(s_{t+1}, \mathbf{V}_{t+1})$. Implicit Taylor's methods have superior stability properties, and one can use a larger step size h. In particular, these methods are effective for stiff systems, while the explicit Taylor's method is not.[22]

As one might expect, both the explicit and implicit Taylor's methods of order d have local truncation error of $O(h^{d+1})$. This means that as the step size $h \to 0$, the local truncation error is proportional to wh^{d+1} for some unknown constant w. To see this, note that a Taylor-series expansion around some point s_t for the explicit Taylor's method (or s_{t+1} for the implicit Taylor's method) involves dropping a term

$$\frac{h^{d+1}}{(d+1)!}\varphi_n^{(d+1)}(\xi_t)$$

for some $\xi_t \in [s_t, s_{t+1}]$ for each t in the grid space. Equation (22) implies

$$\frac{V_{t+1}^n - V_t^n}{h} = g_n(s_t, \mathbf{V}_t) + \frac{h}{2}g_n'(s_t, \mathbf{V}_t) + \cdots + \frac{h^{d-1}}{d!}g_n^{(d-1)}(s_t, \mathbf{V}_t)$$

and, assuming $\varphi_n(\cdot)$ is $\mathbb{C}^{d+1}$ over $[\underline{v}, \overline{s}]$ and given

$$\varphi_n^{(d)}(s_t) = g_n^{(d-1)}\left[s_t, \boldsymbol{\varphi}(s_t)\right],$$

then equation (23) implies

$$\frac{\varphi_n(s_{t+1}) - \varphi_n(s_t)}{h} = g_n[s_t, \boldsymbol{\varphi}(s_t)] + \frac{h}{2}g_n'[s_t, \boldsymbol{\varphi}(s_t)] + \cdots + \frac{h^{d-1}}{d!}g_n^{(d-1)}[s_t, \boldsymbol{\varphi}(s_t)] + O(h^d).$$

[22] In short, a system is referred to as *stiff* when its candidate solutions are sensitive to small changes in the chosen step size. Hairer and Wanner [21] have noted that explicit methods do not work well on stiff problems: stability rather than accuracy govern the choice of step size.

Each step in Taylor's method incurs local truncation error $O(h^{d+1})$ and we require $O(h^{-1})$ steps, so the global truncation error is

$$O(h^d) = O(h^{-1}) \times O(h^{d+1}).$$

While Taylor's methods are attractive in that, given a step size h, the truncation error can be reduced by using higher-order methods.[23] However, Taylor's methods of higher orders require the computation (and evaluation) of higher-order derivatives. One way to avoid this, but still maintain the relationship between order of the method and truncation error, is to use Runge–Kutta methods, which are the most commonly used methods in practice. A further advantage of Runge–Kutta methods is that they are based on a simple formula which considers where the solution is going, but checks that this happens and makes a correction if needed. Finally, Runge–Kutta methods achieve a higher rate of convergence with fewer calculations per step.

Runge–Kutta methods are classified by their order, which corresponds to the order of their (global) truncation error. A Runge–Kutta method of order d has local truncation error of $O(h^{d+1})$ and global truncation error of $O(h^d)$. Here, we present only the classical Runge–Kutta method which is of order four, and use this as a point of discussion for a number of extensions that researchers have developed. Specifically, the fourth-order Runge–Kutta method approximates

$$V_{t+1}^n = V_t^n + \frac{h}{6}\left(z_1^n + 2z_2^n + 2z_3^n + z_4^n\right)$$

where

$$\begin{aligned}
z_1^n &= g_n(s_t, \mathbf{V}_t),\\
z_2^n &= g_n\left(s_t + \frac{h}{2}, \mathbf{V}_t + \frac{h}{2}z_1^n\right),\\
z_3^n &= g_n\left(s_t + \frac{h}{2}, \mathbf{V}_t + \frac{h}{2}z_2^n\right),\\
\text{and}\\
z_4^n &= g_n\left(s_t + \frac{h}{2}, \mathbf{V}_t + \frac{h}{2}z_3^n\right).
\end{aligned}$$

For brevity, we have exploited our notation using

$$\mathbf{V}_t + \frac{h}{2}z_i \equiv \left(V_{1,t} + \frac{h}{2}z_i, V_{2,t} + \frac{h}{2}z_i, \ldots, V_{N,t} + \frac{h}{2}z_i\right).$$

Many different modifications or extensions of Runge–Kutta methods exist. In the classical Runge–Kutta method presented above, the step size is equal between all grid points. Many extensions (e.g., the Runge–Kutta–Fehlberg method and the Dormand–Prince method) allow for an adaptive step size by varying the number and position of steps to ensure that truncation error is below some bound. The classical Runge–Kutta method presented above also uses only information at s_t to compute s_{t+1}: methods with this feature are referred to as *one-step* methods. Methods

[23] Of course, it is possible for a low-order method with a small step size to outperform an high-order method with larger step size, at least in terms of truncation error.

that use m (greater than one) grid points to approximate a function at the next point are referred to as *multistep* methods (e.g., the Adams–Bashforth and the Adams–Moulton methods). Like the Taylor's methods described above, there are also explicit and implicit Runge–Kutta methods. Explicit methods approximate V^n_{t+1} using only previously-determined values $\mathbf{V}_t$ (for mulitstep methods it may involve $\mathbf{V}_{t-1}$, $\mathbf{V}_{t-2}$, etc.). Implicit methods involve solving a system of nonlinear equations which have $\mathbf{V}_{t+1}$ on both sides of each equation (at least in our asymmetric first-price auctions these terms are present in each equation as each bidder's inverse-bid function depends on the inverse-bidding strategies of all other players). In general, implicit methods are used to solve stiff systems of equations as they are more stable than explicit methods.

In general, these methods involve a trade-off between the rate of convergence and the amount of calculation per step. A higher rate of convergence will allow for a larger choice of h and fewer total iterations.

3.2. Projection Methods

An alternative to the shooting algorithms described in the previous subsection are projection methods. A projection method is a general strategy of approximating a true, but unknown, function by a finite number of approximating functions. That is, the true solution is approximated by a finite combination of simple, known functions. For economists, projection methods are, perhaps, more intuitive than the other approaches described above. Specifically, a researcher would first choose a basis to approximate the solutions to each inverse-bid function. The full basis for the space of candidate solutions should be rich (flexible) enough to approximate any function relevant to the problem (which will be represented and approximated as a linear combination of basis functions). This choice specifies the structure of the approximation. The researcher would then fix the flexibility of the approximation by deciding how many basis elements to include. In short, the researcher must fix the order of the approximation. This transforms an infinite-dimensional problem into a finite-dimensional one, where only the coefficients of the basis functions need then to be found. Generally, the only "correct" choice is to use an approximation of infinite order. If the choice of basis is good, then higher orders will yield better approximations. The researcher must also decide on an appropriate residual function to evaluate how closely the approximation represents the true solution. The goal of projection methods is to find a set of coefficients which make some norm of the residual function as close to zero as possible or solves some projection using test functions. Obtaining these coefficients involves solving a set of nonlinear, simultaneous equations or solving a minimization problem. After this has been accomplished, the researcher can verify the quality of the candidate solution and choose either to increase the order of the approximation or, failing that, to begin with a different basis.

Projection methods provide approximate solutions in the form of linear combinations of continuous functions. Some families of projection methods are known by their method of approximation. Spectral methods use bases where each element is nonzero almost everywhere, as with trigonometric bases and orthogonal polynomials. Specifically, Judd [28] has advocated using orthogonal polynomials instead of trigonometric bases because solutions to economics problems generally are not periodic in nature: periodic approximations to nonperiodic functions require many terms to achieve accuracy.

In the case of an asymmetric first-price auction problem, consider approximating each inverse-bid function by a truncated series expansion

$$\hat{\varphi}_n(s) = \sum_{k=0}^{K} \alpha_{n,k}\mathbb{P}_k(s),\ \ s \in [\overline{v}, \overline{s}],\ \ n = 1, 2, \ldots, N$$

where $\mathbb{P}_k(s)$ is some basis functions (which are typically chosen to be polynomials) and the $\alpha_{n,k}$s are referred to as the *spectral coefficients*. Spectral methods often converge exponentially as the order of the approximating polynomial increases. In the finite-element method, (non-overlapping) subdomains are constructed over the domain of interest based on piecewise polynomial interpolation. For the asymmetric first-price auction problem introduced above, consider partitioning the interval $[\underline{v}, \bar{s}]$ into $(T + 1)$ regions, then the inverse-bid function for player n can be approximated by

$$\hat{\varphi}_n(s) = \sum_{t=0}^{T} \alpha_{n,t}\phi_t(s), \ n = 1, 2, \ldots, N$$

where $\phi_t(s)$ is some basis function (for example, piecewise linear polynomials, cubic spline, etc.) and $\alpha_{n,t}$s are now bidder-specific coefficients for subinterval t. As such, finite-element methods use basis functions where each element has a small support, typically involving piecewise functions that are nonzero almost everywhere on the subdomain. Thus, spectral methods use *global* basis functions in which each term in the series is a polynomial (and the last term is of high order). Finite-element methods use *local* basis functions on the subdomains (of fixed order), which are then aggregated to approximate the function(s) over the full domain.

For economists, perhaps the most intuitive spectral method is that of least-squares. Consider, again, the set of N first-order ODEs for which a representative equation for bidder n was given by system (13), which we shall express as

$$\begin{aligned} \varphi_1'(s) &= g_1\left[s, \boldsymbol{\varphi}(s)\right] \\ &\vdots \\ \varphi_N'(s) &= g_N\left[s, \boldsymbol{\varphi}(s)\right]. \end{aligned}$$

Under the spectral method considered above, each inverse-bid function is approximated by a truncated series expansion of some basis functions. The problem is to estimate $\bar{s}$ as well as the $\alpha_{n,k}$s for all $n = 1, 2, \ldots, N$ and $k = 0, 1, \ldots, K$. Consider selecting a large number T of grid points from the interval $[\underline{v}, \bar{s}]$. The system can be evaluated at each grid point and the parameters can be chosen to minimize the following criterion function:

$$H(\bar{s}, \boldsymbol{\alpha}) = \sum_{n=1}^{N} \sum_{t=1}^{T} \left(\hat{\varphi}_n'(s_t) - g_n\left[s_t, \hat{\boldsymbol{\varphi}}(s_t)\right]\right)^2$$

where $\boldsymbol{\alpha}$ denotes a vector that collects the $N \times (K + 1)$ coefficients of the polynomials. To economists, the least-squares approach is compelling: we have reduced the problem of solving a functional equation to solving a nonlinear minimization problem, a problem with which we have considerable experience. In many problems, boundary conditions can be satisfied by the choice of the basis. A challenge in the context of an asymmetric first-price auction is the presence of the free boundary. In this formulation, the boundary conditions must enter the objective function directly or be imposed as constraints, which leads to a constrained optimization problem. We shall investigate and formalize these alternatives below when summarizing the previous research. The method of least squares is an attractive way of approximating these solutions. In fact, some have argued that it is a safe way of generating approximations that depend nonlinearly on the unknowns; see, for example, Boyd [6].

In contrast to the method of least squares, collocation (pseudospectral) methods work under the assumption that the solution can be represented by a candidate family of functions (typically polynomials); collocation involves selecting a candidate which solves the system exactly at a set of points on the interval of interest. These points are referred to as the *collocation points*. Specifically, each component of the residual is set to equal zero exactly by choosing the number of collocation points (including the boundary conditions) T to equal the number of unknown coefficients $N(K+1)$. Of course, the common high bid $\overline{s}$ is also unknown, so there are $N \times (K + 1) + 1$ unknowns in total. Collocation is akin to interpolating a known function by requiring the approximation to coincide with the true function at a collection of points, so it is also referred to as an *interpolating spectral* method. Substituting the approximations for the true, but unknown, inverse-bid functions and computing the gradient yields a system of equations which must be solved for the spectral coefficients. Orthogonal collocation involves choosing the grid points to correspond with the K zeros of the K^{th} orthogonal polynomial basis element and the basis elements are orthogonal with respect to the inner product. Provided the residual is smooth in the bids s, the Chebyshev interpolation theorem says that these zero conditions will force the residual to be close to zero for all $s \in [\underline{v}, \overline{s}]$. Likewise, the optimality of Chebyshev interpolation also says that if one is going to use collocation, then these are the best possible points to use.

Solving for the spectral coefficients requires either a minimization algorithm or a nonlinear algebraic equation solver. If the system of equations is overidentified, or if one is minimizing the "sum of squared residuals," then a nonlinear least-squares algorithm may be used. Good initial guesses are important because projection methods involve either a system of nonlinear equations or optimizing a nonlinear objective. Judd [28] has advocated a two-stage approach. In the first stage, the method of least squares is used, along with a loose convergence criterion, to compute quickly a low-quality approximation; in the second stage, this approximation is used as an initial guess for a projection method involving a higher-order approximation. Sometimes, the finite-dimensional problem generated by a projection method will not have a solution, even when the original problem does have a solution. If solving the system for a particular basis and a particular order K is proving difficult, then using another basis or order may resolve the issue. Regardless, one way to ensure existence of a solution is to construct a least-squares objective (which may overidentify the problem) as an approximation is assured as long as the objective(s) are continuous and optimization methods are reliable.

4. Previous Research concerning Numerical Solutions

In this section, we discuss research that either directly or indirectly contributed to improving computational methods to solve for bidding strategies at asymmetric first-price auctions.

4.1. Marshall, Meurer, Richard, and Stromquist (1994)

The first researchers to propose using numerical algorithms to solve for the equilibrium (inverse-) bid functions at asymmetric auctions were Marshall et al. [41]. Marshall et al. investigated a model in which all bidders draw valuations from a uniform distribution. An asymmetry obtained because there existed two coalitions at the auction, with a different number of bidders belonging to each (the coalitions were of different size).[24] Thus, as we described above, the

[24]Marshall et al. discussed how their research could be extended to a model in which two-types of bidders draw valuations from any two arbitrary distributions. The presentation is similar to the section "Bidders from Different Urns" that we presented earlier which involved two bidders, although they allowed for more than one bidder from each class.

model of Marshall et al. simplifies to an asymmetric auction with two bidders who each draw valuations from a different power distribution.[25]

Marshall et al. applied l'Hôpital's rule to the first-order conditions to derive

$$\lim_{s \to 0^+} \varphi_k'(s) = \frac{n_k + 1}{n_k}$$

where n_k is the number of bidders in coalition k and $\varphi_k(\cdot)$ is coalition k's inverse-bid function. They found that successive (higher) derivatives of $\varphi_k(\cdot)$ are zero at $\underline{v}$. Because of this, forward numerical integration produces a linear solution described by

$$\varphi_k(s) = \frac{n_k + 1}{n_k} s.$$

This "nuisance" solution is wrong because it does not satisfy the appropriate right-boundary condition that all coalitions submit the same bid $\bar{s}$ when they have the highest valuation $\bar{v}$. This fact motivated them to use a backward shooting algorithm, like the ones we described above, in which they assumed a "terminal" (right-boundary) point $\bar{s}$, integrate backwards, and then used the "initial" (left-boundary) condition that all coalitions bid $\underline{v}$ when they have the lowest valuation $\underline{v}$ to check the validity of the solution. What drives the shooting algorithm is the notion that the assumed value of $\bar{s}$ needs to be increased or decreased at a given iteration based on the value that the candidate solution takes at $\underline{v}$.

In practice, for stability reasons, Marshall et al. advocated normalizing the inverse-bid functions by the bid at which they are being evaluated; that is, solving for

$$\delta_k(s) = \frac{\varphi_k(s)}{s}$$

rather than $\varphi_k(s)$. They approximated the $\{\delta_k(s)\}_{k=1}^{2}$ values by Taylor-series expansions of order p (chosen to be five) around each point $s_t \in [\underline{v}, \bar{s}]$ where s_t represents one of T equal to $10,000$ grid points. In doing so, they used analytic functions for the successive derivatives of the transformed inverse-bid functions to avoid inaccuracy of high-order numerical derivatives. Their approach requires an efficient algorithm for evaluating the Taylor-series expansions and a reasonable convergence criterion. Marshall et al. considered the approximate solution valid when

$$\frac{1}{2} \sum_{k=1}^{2} \left[\delta_k(\underline{v}) - \frac{n_k + 1}{n_k} \right] \leq \varepsilon^2$$

where ε was chosen to be of the order 10^{-5} to 10^{-8}. They adapted this convergence criterion somewhat to deal with cases involving greater numerical instability in the neighborhood of $\underline{v}$; see Appendix B of their paper.

While Marshall et al. only considered the case of two coalitions competing at auction with bidders in each coalition receiving uniform draws, they suggested that their approach could be adapted to general distributions. Their uniform/power distributional assumption was convenient in that there was no need to evaluate nonlinear cumulative distribution (and probability density)

[25] While Marshall et al. realized that a closed-form solution existed under these conditions (in fact, they characterized this solution partially in their Appendix A), they proposed numerical methods as being applicable to a general class of problems noting that this (power distribuiton) case "seems to be the exception rather than the rule."

functions at each inverse-bid function because these terms canceled each other out in the first-order conditions. Extending their approach to general distributions means that terms involving $F_n[\varphi_n(s)]$ enter the first-order conditions. A practical difficulty with this is that Taylor-series expansions for these functions must be included in the algorithm via implementation of an appropriate chain rule—see Appendix C of Marshall et al. [41].

4.2. Bajari (2001)

Bajari [3] observed that bid-rigging (collusive bidding) was a serious problem at procurement auctions. He sought to provide empirical researchers with a way to assess whether observed bidding behavior was consistent with competitive bidding. However, empirical researchers have argued that models of bidding should have asymmetric bidders; for example, Bajari [2] found that 75 percent of the highway construction contracts he observed were awarded to the firm located closest to the project—clearly location was an important source of asymmetry.[26] To deal with this, Bajari proposed algorithms for computing equilibrium (inverse-) bid functions when asymmetric bidders competed at auction. In doing so, his research extended that of Marshall et al. [41] by allowing for N bidders who each draw valuations from different, arbitrary distributions, provided that some regularity conditions are satisfied. For us, this research is relevant because Bajari [3] proposed three computational algorithms to compute equilibria in these models, although we present only two of these.[27] While Bajari considered a procurement setting in his research, we maintain our discussion of first-price asymmetric auctions.

Bajari's first algorithm is essentially a straightforward application of the shooting algorithm we discussed above.[28] He provided a formal proof that the convergence-divergence behavior of the inverse-bid functions and the known boundary $\underline{v}$ ($\bar{c}$ in his case) at a first-price (procurement) auction can be used to adjust the starting, unknown bid $\bar{s}$ ($\underline{b}$); see Lemmata 7 and 8 in Appendix B of his paper which informed our earlier discussion of this procedure. Bajari suggested that "standard methods in the numerical analysis of ordinary differential equations" can be used to solve the system and adopted a Runge–Kutta algorithm.[29,30]

Under Bajari's third algorithm, the inverse-bid functions are approximated by flexible functional forms. Specifically, he assumed that the inverse-bid function for bidder n can be represented by a linear combination of ordinary polynomials. Although Bajari considered the pro-

[26]It would also be reasonable to think that the firms have different managerial ability, capacity constraints, capabilities, and so forth.

[27]We do not discuss the second method Bajari proposed which involves assuming initially that firms bid their cost and then solving iteratively for the best-response (at each bid) of rival firms to this strategy profile until each firm is best-responding to all other firms' strategies.

[28]Although published six years later, Bajari [3] credits Li and Riley [37] (although his reference is to a paper by Riley and Li with the same title) as generalizing this shooting algorithm. While Marshall et al. [41] used Taylor-series expansions to integrate backwards, Li and Riley [37] used the Bulirsch–Stoer method. The Bulirsch–Stoer method uses a modified midpoint method in integration. A large interval is spanned by different sequences of fine substeps and extrapolation uses rational functions; see Appendix B of Li and Riley [37]. Li and Riley also introduced a bisection algorithm to speed-up search for the high bid $\bar{s}$.

[29]We thank Pat Bajari for providing us with his original computer program and a user guide. While he did not advocate explicitly a method for approximating the solution, the program shows that a Runge–Kutta algorithm was used, but he warned users in his accompanying guide that results are sensitive to the tolerance criterion, something we have found as well.

[30]Dalkir, Logan, and Masson [13] also used a (fourth-order) explicit Runge–Kutta method. They considered a procurement environment in which bidders each draw a cost from a uniform distribution. The asymmetry enters their model when the effects of a merged firm are introduced.

curement case, here we consider the the case of sale, so

$$\hat{\varphi}_n(s) = \overline{s} - \sum_{k=0}^{K} \alpha_{n,k} (\overline{s} - s)^k, \; n = 1, 2, \ldots, N. \tag{24}$$

Note that equation (12) implies that the first-order condition for bidder n can be expressed as

$$1 = [\varphi_n(s) - s] \sum_{m \neq n} \frac{f_m[\varphi_m(s)]}{F_m[\varphi_m(s)]} \varphi'_m(s).$$

Define $G_n(s; \overline{s}, \boldsymbol{\alpha})$ as

$$G_n(s; \overline{s}, \boldsymbol{\alpha}) \equiv 1 - [\hat{\varphi}_n(s) - s] \sum_{m \neq n} \frac{f_m[\hat{\varphi}_m(s)]}{F_m[\hat{\varphi}_m(s)]} \varphi'_m(s) \tag{25}$$

where $\boldsymbol{\alpha}$ denotes a vector that collects the $N \times (K+1)$ coefficients of the polynomials. In an exact solution, $G_n(s; \overline{s}, \boldsymbol{\alpha})$ should equal zero for all bidders and at any bid $s \in [\underline{v}, \overline{s}]$. In addition, an exact solution must satisfy the left-boundary condition

$$\varphi_n(\underline{v}) = \underline{v}$$

and the right-boundary condition

$$\varphi_n(\overline{s}) = \overline{v}$$

for each bidder n.

Because the common high bid $\overline{s}$ is *a priori* unknown, there are $[N \times (K + 1) + 1]$ unknowns which must be found. Bajari proposed selecting a large number T of grid points uniformly-spaced over the $[\underline{v}, \overline{s}]$ interval and choosing these unknown parameters to minimize

$$H(\overline{s}, \boldsymbol{\alpha}) \equiv \sum_{n=1}^{N} \sum_{t=1}^{T} [G_n(s_t; \overline{s}, \boldsymbol{\alpha})]^2 + \sum_{n=1}^{N} \left[\hat{\varphi}_n(\underline{v}) - \underline{v}\right]^2 + \sum_{n=1}^{N} [\hat{\varphi}_n(\overline{s}) - \overline{v}]^2 . \tag{26}$$

If all the first-order conditions as well as the boundary conditions are satisfied, then the objective $H(\overline{s}, \boldsymbol{\alpha})$ will equal zero. Bajari reported finding accurate solutions (to five or more significant digits) when only third- or fourth-order polynomials were used. In practice, Bajari chose K to equal five and used a nonlinear least-squares algorithm to select $\overline{s}$ and $\boldsymbol{\alpha}$ by minimizing a modified version of equation (26)

$$\tilde{H}(\overline{s}, \boldsymbol{\alpha}) \equiv \sum_{n=1}^{N} \sum_{t=1}^{T} [G_n(s_t; \overline{s}, \boldsymbol{\alpha})]^2 + T \sum_{n=1}^{N} \left[\hat{\varphi}_n(\underline{v}) - \underline{v}\right]^2 + T \sum_{n=1}^{N} [\hat{\varphi}_n(\overline{s}) - \overline{v}]^2$$

which adds weight to the boundary conditions.

An advantage of the polynomial approximation approach is that it is extremely fast. This is particularly important for econometricians who might need to recompute the solution within some estimation routine at each iteration. Researchers who wish to simulate dynamic models involving asymmetric auctions can also benefit from using this algorithm. However, to realize this gain in speed relative to shooting-based algorithms, the user must provide good starting values for the algorithm.

4.3. Fibich and Gavious (2003)

Fibich and Gavious [14] recognized that, while it is impossible to solve for the equilibrium bid functions at an asymmetric first-price auction analytically, it is well known how to solve for the closed-form solution to a symmetric first-price auction. The authors suggested using perturbation analysis to calculate an *explicit approximation* to the asymmetric first-price solution to gain insight into revenue and efficiency questions in settings in which the asymmetry is small. Specifically, Fibich and Gavious defined the average distribution among N bidders at a valuation v as

$$F_{\text{avg}}(v) \equiv \frac{1}{N}\sum_{n=1}^{N} F_n(v).$$

Let ε be a parameter that measures the level of asymmetry which is defined as

$$\varepsilon = \max_{n\in[1,N]} \max_{v\in[\underline{v},\overline{v}]} |F_n(v) - F_{\text{avg}}(v)|.$$

Given these two definitions, auxiliary functions $A_n(v)$ can be constructed as

$$A_n(v) = \frac{F_n(v) - F_{\text{avg}}(v)}{\varepsilon}.$$

These auxiliary functions have the property that

$$\sum_{n=1}^{N} A_n(v) = 0$$

and, for all $v \in [\underline{v}, \overline{v}]$

$$|A_n(v)| \leq 1, \ n = 1, \ldots, N.$$

Since valuations are drawn from a common, compact support $[\underline{v}, \overline{v}]$,

$$A_n(\underline{v}) = 0, \ n = 1, \ldots, N,$$

and

$$A_n(\overline{v}) = 0, \ n = 1, \ldots, N,$$

as the average value of the distribution equals the value of each distribution at the endpoints of the interval.

They noted that, by construction,

$$F_n(v) = F_{\text{avg}}(v) + \varepsilon A_n(v), \ n = 1, \ldots, N$$

where $\varepsilon A_n(\cdot)$ represents the perturbation from the average distribution for player n. Fibich and Gavious proved that, within this framework, the equilibrium bid functions can be solved for explicitly as:[31]

$$\sigma_n(v) = \sigma_{\text{avg}}(v) + \varepsilon E_n[v, A_n(v)] + O(\varepsilon^2)$$

[31] Unlike the numerical methods we have described, which require solving for the inverse-bid functions, Fibich and Gavious characterized the bid functions directly; see Propositions 1 and 3 of their paper.

where $\sigma_{\text{avg}}(\cdot)$ is the equilibrium bid function at a first-price auction when all bidders draw valuations from $F_{\text{avg}}(\cdot)$ and

$$E_n[v, A_n(v)] = \frac{-(N-1)}{F_{\text{avg}}^{N-1}(v)} \left[\int_{\underline{v}}^{v} F_{\text{avg}}^{N-1}(u)\, \mathrm{d}u\right]^N \int_{v}^{\bar{v}} \frac{1}{\left[\int F_{\text{avg}}^{N-1}(u)\, \mathrm{d}u\right]^{N-1}} \frac{\mathrm{d}\left[\frac{A_n(t)}{F_{\text{avg}}(t)}\right]}{\mathrm{d}t}\, \mathrm{d}t.$$

The perturbation approach of Fibich and Gavious is attractive because explicit expressions for the equilibrium bids can be derived; these allow researchers to extend theoretical results when asymmetries are sufficiently small, provided certain conditions (typically on the auxiliary functions) are satisfied.[32]

While the perturbation approach has been shown to be theoretically useful in deriving explicit approximations, the user sacrifices "exactness." Another concern is the size ε be for approximations to be valid. To consider this, Fibich and Gavious used a shooting algorithm to compare the explicit approximations to those computed numerically, which they referred to as the *exact solutions*. They demonstrated that the approximations are quite good when ε is less than 0.25; however, the approximations of the bidding functions are visually distinct for ε greater than 0.50. Thus, the perturbation approach is helpful to researchers using numerical methods in that it can inform the researcher concerning a good initial guess which will speed-up convergence by reducing the number of required iterations. Specifically, algorithms built-off shooting algorithms require an initial guess for the *a priori* unknown high bid $\bar{s}$. The results of Fibich and Gavious suggest the following initial guess:

$$\bar{s} \approx \bar{v} - \int_{\underline{v}}^{\bar{v}} F_{\text{avg}}^{N-1}(v)\, \mathrm{d}v.$$

While this may seem like a minor point, a common difficulty encountered when using shooting algorithms to solve boundary-value problems is that obtaining convergence in an efficient way (for example, by using Newton-type methods) requires accurate initial estimates of (in our case) $\bar{s}$.

4.4. Gayle and Richard (2008)

Gayle and Richard [18] presented, perhaps, the most practical contribution to field. They generalized the backwards shooting algorithm of Marshall et al. [41] to allow for N bidders who draw valuations (costs) from four commonly-used distributions. Gayle and Richard also provided a user-friendly computer program, which is available free to researchers.[33] Gayle and Richard claimed, too, that researchers could specify arbitrary (parametric, semiparametric, or nonparametric) distributions which their program could accommodate by constructing local Taylor-series expansions of (the inverse of) the (truncated) distributions automatically.

Gayle and Richard adjusted the Marshall et al. algorithm in two ways: first, they approximated functions of the inverse-bid functions; second, they incorporated a minimization routine

[32] Fibich and Gavious [14] extended results concerning expected revenue and the probability of inefficiency obtaining. Typically, the results hold within $O(\varepsilon^2)$ accuracy. Fibich et al. [17] extended the Revenue Equivalence Theorem using a perturbation approach.

[33] Gayle and Richard also cited the working paper of Li and Riley [37] (as Riley and Li) who introduced BIDCOMP^2 software which can compute bid functions under restricted scenarios.

to search for the unknown high bid $\bar{s}$. Recall that equation (12) can be written as

$$1 = [\varphi_n(s) - s] \sum_{m \neq n} \frac{f_m[\varphi_m(s)]}{F_m[\varphi_m(s)]} \varphi_m'(s).$$

Gayle and Richard defined

$$\ell_n(s) \equiv F_n[\varphi_n(s)]$$

and rewrote this first-order condition for player n as

$$1 = \left(F_n^{-1}[\ell_n(s)] - s \right) \sum_{m \neq n} \frac{\ell_n'(s)}{\ell_n(s)}$$

where we have applied the chain rule to derive $\ell_n'(s)$. Rather than approximate the inverse-bid functions, they used Taylor-series expansions of $F_n^{-1}(\cdot)$ and generated expansions of $\ell_n(s)$, $\left(F_n^{-1}[\ell_n(s)] - s \right)$, as well as $[\ell_n'(s)/\ell_n(s)]$ for each n; that is, each of these three components was approximated by a Taylor-series expansion. The coefficients of each expansion are computed recursively with the $[\ell_n'(s)/\ell_n(s)]$ term providing the link between coefficients of one order and those of the next. Once the piecewise approximation is finished at one point s_t, the algorithm proceeds (backwards) to s_{t-1} which equals $(s_t - \Delta s)$ where Δs is the uniform step size. Rather than specify a convergence criteria, Gayle and Richard chose the high bid $\bar{s}$ by solving the following minimization problem:

$$\min_{\bar{s} \in [r_0, \bar{v}]} \sum_{n=1}^{N} [\ell_n(r_0 | \bar{s}) - F_n(r_0)]^2$$

where r_0 denotes a potential reserve price and equals $\underline{v}$ when no reservation price exists.

The program that Gayle and Richard provided computes the bid functions without the presence of a reserve price as well as with the optimal reserve price. The algorithm also computes expected profits, probabilities of winning, expected revenues to the seller, and the probability the seller receives no bids in the presence of a reserve price. These statistics are computed for second-price auctions as well. Many of these values are expressed as the solution to integrals which the authors computed numerically via univariate quadrature: they used the extended Simpson's rule.

The approach of Gayle and Richard relies on equally-spaced subdivisions of the valuation support. They warned that "occasional pathologies" would require adaptive selection of the step size, and addressed this issue by increasing the number of points in the grid space as needed, showing in an example that, for a given order Taylor-series expansion, increasing the number of grid points (creating a finer grid) reduced the root mean squared error (RMSE) which was computed using each player's first-order condition. The reduction of RMSE was marginal, but the cost (as measured by the increase in computational time) was substantial.[34] This suggests there is little to gain by considering a finer grid within the algorithm. Furthermore, for a given number of grid points, higher-order Taylor-series expansions did not reduce the RMSE but did

[34] In their example, in which just two bidders were considered, each of whom drew valuations from asymmetric Weibull distributions, and a fifth-order Taylor-series expansion was used, it took 37 seconds to solve the problem with 500 grid points, and 202 seconds to solve the problem with 2,000 grid points, while the RMSE fell by only 0.45 percent. Instead of creating a finer grid, Li and Riley [37] used a step size that depended on the allowed error tolerance and used more steps for problems with more curvature in the inverse-bid functions.

increase the computational time.[35] In fact, Gayle and Richard cautioned readers that "an order of approximation that is too high can lead to significant numerical pathologies." This conclusion may obtain because the computation of higher order derivatives is not numerically stable and is prone to error.

4.5. Hubbard and Paarsch (2009)

One downside to the shooting-based algorithms of Marshall et al. [41], Bajari [3] (his first method), and Gayle and Richard [18] is that they take a long time to compute. Higher accuracy clearly comes at the cost of increased computational time. In fact, Gayle and Richard warned that, if the model involves bidders of many different types, then "the potential computational time increases significantly." Computation time is of critical importance to empirical researchers who often need to solve for the equilibrium (inverse-) bid functions for each candidate vector of parameters when estimating the underlying distributions of private values, which may be conditioned on covariates as well. However, empirical researchers also need to be concerned with the error associated with computing the equilibrium (inverse-) bid functions which might bias estimates. Likewise, if researchers need to simulate dynamic games that require computing the inverse-bid functions at each period, then speed is crucial as this may require solving for the inverse-bid functions thousands of times. For example, Saini [54] solved for a Markov Perfect Bayesian Equilibrium (MPBE) in a dynamic, infinite-horizon, procurement auction in which asymmetries obtain endogenously because of capacity constraints and utilization. Likewise, Saini [55] allowed for entry and exit as well as for firms to invest in capacity in a dynamic oligopoly model in which firms compete at auction each period to consider the evolution of market structure as well as optimality and efficiency comparisons between first-price and second-price auctions in a dynamic setting. This required him to solve for the inverse-bid functions for each firm, at each state (described by each firm's capacity), at each iteration in computing the MPBE.

Typically, with spectral methods, researchers use either the Fourier basis functions (usually for periodic problems) or orthogonal polynomials. Judd [28] has warned against the choice of ordinary polynomials as a basis. While the Stone–Weierstraß theorem guarantees that ordinary polynomials are complete in the L_1 norm, given our interest in bounded, measurable functions on a compact set, this does not help when a least-squares objective is used. They will not be orthogonal in any natural inner product on $\mathbb{R}^+$. Furthermore, these polynomials may not be a good choice for a basis given their behavior is too similar; for example, they are all monotonically increasing and positive on $\mathbb{R}^+$. Finally, they can vary dramatically over an interval $[\underline{v}, \overline{s}]$.

For example, see figure 3 which depicts several examples of Chebyshev polynomials, which are used in auctions because they are defined on a closed interval, while in figure 4, we depict the corresponding standard polynomials. Clearly, the Chebyshev polynomials will do better in this problem.

Hubbard and Paarsch [23] modified the polynomial approximation approach of Bajari [3] in three ways: first, instead of regular polynomials, they employed Chebyshev polynomials, which are orthogonal polynomials and thus more stable numerically. In addition to an orthogonal basis, Hubbard and Paarsch recommended coupling this choice with a grid defined by the roots of the

[35] The lowest computation time involved a second-order expansion with 500 grid points which took 18 seconds. Holding fixed the number of grid points, the RMSE was constant for all orders of Taylor-series expansions between two and five which is what Gayle and Richard recommended.

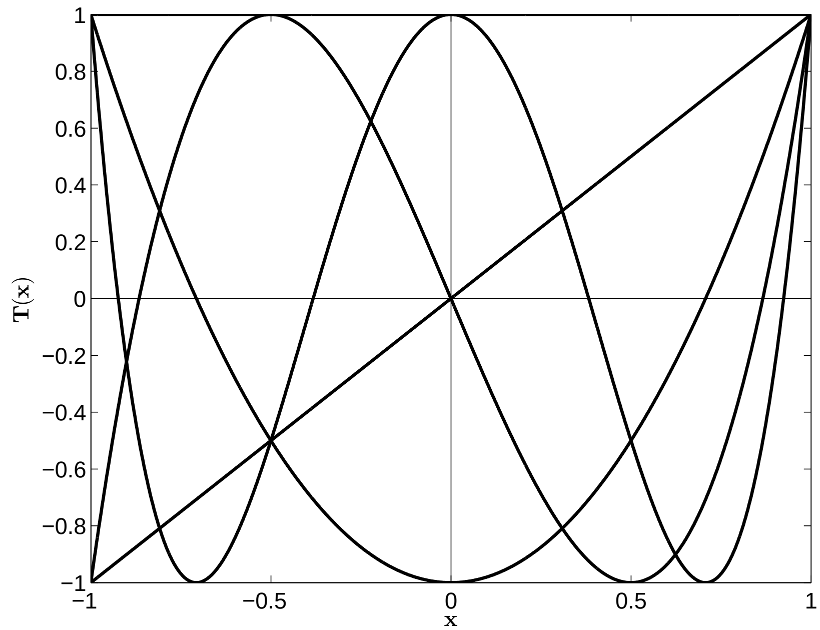

Figure 3: Some Examples of Chebyshev Polynomials

T^{th} order Chebyshev polynomial—the Chebyshev nodes—which can be computed as

$$x_t = \cos\left[\frac{\pi(t-1)}{T}\right], \quad t = 1, \ldots, T.$$

The Chebyshev nodes lie on the interval $[-1, 1]$, the domain of the Chebyshev polynomials. The points $\{s_t\}_{t=1}^{T}$ are found via the following transformation:

$$s_t = \frac{\overline{s} + \underline{v} + (\overline{s} - \underline{v})x_t}{2}.$$

Second, Hubbard and Paarsch cast the problem within the Mathematical Programs with Equilibrium Constraints (MPEC) approach advocated by Su and Judd [57]. (For more on the MPEC approach, see Luo et al. [39].) Specifically, they used the MPEC approach to discipline the estimated Chebyshev coefficients in the approximations so that the first-order conditions defining the inverse equilibrium-bid functions are approximately satisfied, subject to constraints that the boundary conditions defining the equilibrium strategies are satisfied. Finally, they imposed monotonicity on candidate solutions. We make explicit the constrained optimization problem below when discussing the work of [26] who have extended this line of research.

4.6. Fibich and Gavish (2011)

While most researchers who have employed shooting-based algorithms to solve asymmetric auctions have noticed their inherent instability, Fibich and Gavish [15] proved that this was

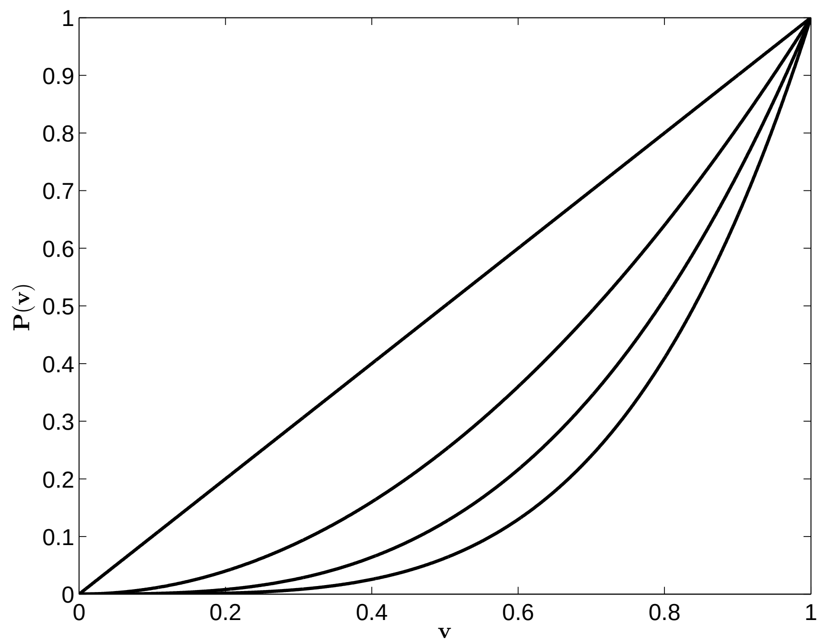

Figure 4: Some Examples of Standard Polynomials

not a "technical issue," but rather an analytic property of backward integration in this setting.[36] Fibich and Gavish demonstrated this instability by first considering a very controlled, and well-understood setting—the symmetric first-price auction. As we noted above, the equilibrium bid function in this case can be solved in closed-form. Fibich and Gavish showed that a solution obtained via backwards integration that involves being ε away from the true high bid (which can also be calculated analytically in a symmetric setting) involves an absolute error that increases monotonically from ε (at the high bid, by construction) to infinity as $v \to \underline{v}$. Furthermore, the sensitivity of the backwards-shooting solution becomes worse as the number of players increases. Given this instability obtains in a symmetric setting, it is not surprising that the results extend to one involving heterogeneous bidders.

Not only did Fibich and Gavish document the problem of instability formally, but they also suggested an alternative approach to solving asymmetric auctions. Specifically, they transformed the free boundary-value problem involving $\bar{s}$ to one in which one player's valuation is written explicitly as a function of the other player's valuation.[37] In doing so, they obtained a standard (though, still nonlinear) boundary-value problem which they proposed solving via fixed-point iteration. For example, in the two-player case, using a change of variables, the system (11), can

[36] For example, [41] as well as [37] made explicit statements concerning the poor behavior of the solutions in the neighborhood of the lower end-point.

[37] Fibich and Gavish noted that the choice of which valuation to use as the independent variable is *ad hoc* and can lead to divergence—see footnote 7 as well as the conclusion of their paper in which they suggested this choice might be worth pursuing in future research.

be rewritten as

$$\frac{\mathrm{d}v_1(v_2)}{\mathrm{d}v_2} = \frac{F_1[v_1(v_2)]f_2(v_2)[v_1(v_2) - s(v_2)]}{f_1[v_1(v_2)]F_2(v_2)[v_2 - s(v_2)]}$$
$$\frac{\mathrm{d}s(v_2)}{\mathrm{d}v_2} = \frac{f_2(v_2)}{F_2(v_2)}[v_1(v_2) - s(v_2)].$$

An advantage of this transformation is that the dependent variable v_2 is known to be in the range $[\underline{v}, \overline{v}]$. In the canonical system of ODEs, the dependent variable s has an unknown support as $\overline{s}$ is unknown *a priori*. Under this transformation, the left-boundary condition can be expressed as

$$v_1(\underline{v}) = s(\underline{v}) = 0,$$

while the known right-boundary condition becomes

$$v_1(\overline{v}) = \overline{v}.$$

Note that $\overline{s}(\overline{v})$ is still unknown *a priori*, but it can be recovered once the system has been solved. Fibich and Gavish suggested solving this system using fixed-point iteration: discretize v_2 and construct an initial guess for the solutions v_1 and s over the grid of v_2 points. Next, solve a modified version of the system above which provides new values for v_1 and s, respectively. Of course, the v_1 and s values feed into the equation determining the other as any modification of the system still involves both variables. Use the new v_1 and s values in the next iteration and continue this procedure. A researcher can incorporate tolerance criteria based on a norm involving the changes in $v_1(v_2)$ and $s(v_2)$ between iterations to determine when to stop cycling through the iterative procedure. Since fixed-point problems can be expressed as root-finding problems, it is perhaps not surprising that Fibich and Gavish also suggested using Newton's method (Newton's iterations) to solve their transformed system. An advantage of this technique is that it can speed up convergence—although Newton's method would be very complicated to implement with more than two bidders.

Fibich and Gavish also investigated some interesting applications of their method to problems that cannot be solved using backwards shooting—even if the shooting algorithm were reliable, simply because it is too time-consuming. In particular, they first presented an example with a large number of heterogeneous bidders to argue that, as N get large (tends to infinity), asymmetric auctions "become symmetric." [4] showed that all symmetric mechanisms, which in equilibrium award objects to a highest signal observer and only have payments conditional on winning, generate the same limiting revenue, which is equal to the expected value of the best-use of the object. Given that Fibich and Gavish found that asymmetric auctions become symmetric, they employed the result of Bali and Jackson to examine the rate at which asymmetric auctions become revenue equivalent as the number of players increases.

4.7. Hubbard, Kirkegaard, and Paarsch (2011)

Hubbard et al. [26] expanded on the suggestions proposed by Hubbard and Paarsch [23] by using economic theory to constrain approximations further and as a guide to determine the quality of the solutions. Of the research discussed so far, theirs relies most on connecting economic theory with numerical analysis and levering this interdependence. Although it is typically impossible to solve the system of differential equations that characterizes equilibrium bidding in closed-form, some properties can be deduced by studying the system at the endpoints, as s approaches $\underline{v}$ or $\overline{s}$. Fibich et al. [16] proved the following properties concerning the high and low types, the first of which follows directly from system (13):

1. $\sum_{m \neq n}(\overline{v} - \overline{s}) f_m(\overline{v}) \varphi'_m(\overline{s}) = 1$ for all $n = 1, 2, \ldots, N$.
2. If $f_n(\underline{v}) \in \mathbb{R}_{++}$ and $\varphi_n(s)$ is differentiable at $s = \underline{v}$ for all $n = 1, 2, \ldots, N$, then $\varphi'_n(\underline{v}) = [N/(N-1)]$.

The second condition generalizes the [41] result as $s \to 0$ in their restricted model. This second condition holds if the probabilty density functions for each bidder are strictly greater than zero and if $\varphi_n(s)$ is differentiable at $\underline{v}$. Given Hubbard et al. represented the inverse-bid functions by (Chebyshev) polynomials, this latter condition will hold in their approximations. Thus, these two properties imply two conditions, in addition to the right-boundary and left-boundary conditions that will characterize each inverse-bid function.

For each bidder, the authors imposed four constraints on the equilibrium inverse-bid functions, which they approximated by Chebyshev polynomials of order K. Specifically, Hubbard et al. [26] approximated the solution to (13) by solving

$$\min_{\{\overline{s}, \boldsymbol{\alpha}\}} \sum_{n=1}^{N} \sum_{t=1}^{T} [G_n(s_t; \overline{s}, \boldsymbol{\alpha})]^2$$

subject to the following conditions for each bidder n:

1. $\varphi_n(\underline{v}) = \underline{v}$
2. $\varphi_n(\overline{s}) = \overline{v}$
3. $\sum_{m \neq n}(\overline{v} - \overline{s}) f_m(\overline{v}) \varphi'_m(\overline{s}) = 1$
4. $\varphi'_n(\underline{v}) = [N/(N-1)]$
5. $\varphi_n(s_{j-1}) \leq \varphi_n(s_j)$ for a uniform grid $j = 2, \ldots, J$

where $G_n(\cdot)$ was defined in equation (25). The last condition is a shape-preservation one: monotonicity is imposed on the solution at a grid of J additional points not considered in the objective function. The MPEC approach is used to discipline the estimated Chebyshev coefficients in the approximations so that the first-order conditions defining the equilibrium inverse-bid functions are approximately satisfied, subject to constraints that the boundary conditions defining the equilibrium strategies are satisfied.

Under this approach there are $4N$ conditions (constraints) in total and TN points that enter the objective function. By comparison, there are $N(K+1)+1$ parameters to be estimated—the parameters in $\boldsymbol{\alpha}$ plus $\overline{s}$. For the number of conditions (boundary and first-order together) to equal the number of unknowns

$$N(T+4) = N(K+1) + 1$$

or

$$(T+4) = (K+1) + \frac{1}{N}. \tag{27}$$

Since at auctions, N weakly exceeds two, and T and K are integers, this equality cannot hold for any (T, K) choice. However, when comparing the $N(K+1)+1$ parameters with the $4N$ conditions, note that, if K equals three and all the conditions are satisfied, then only one degree of freedom remains. One criticism of the polynomial approximation approach (and projection-based methods, in general) is that it works well, if the practitioner has a good initial guess. When K equals three, the researcher obtains an initial guess that already satisfies some theoretical properties at essentially no cost because there is only one free parameter, $\overline{s}$, to minimize the nonlinear least-squares objective.

This approach is related to the spectral methods used to solve partial differential equations—see the discussion above and Gottlieb and Orszag [19] for a book-length treatment. Remember that, under collocation methods, it is assumed that the solution can be represented by a candidate approximation, typically a polynomial; a solution is selected that solves the system *exactly* at a set of (collocation) points over the interval of interest. Because equality (27) cannot hold, collocation is infeasible in this case, but the MPEC-based approach can be thought of as an hybrid between collocation and least-squares as some constraints are explicitly imposed, leading to a constrained nonlinear optimization problem. It will be impossible to make all residual terms equal to zero. In other words, the fit is necessarily imperfect in a quantitative sense.

Hubbard et al. showed that there may be qualitative inadequacies of this approach as well when the order of the polynomial used in the approximation is too small. Specifically, the authors levered theoretical results from Kirkegaard [31] to evaluate the quality of an approximation. Kirkegaard proved that, in the two bidder case, if $F_1(v)$ crosses $F_2(v)$, then the equilibrium bid functions must cross as well. Under certain conditions, he determined the exact number of times the bid functions will cross. Let

$$P_{n,m}(v) = \frac{F_m(v)}{F_n(v)},\ v \in (\underline{v}, \overline{v}]$$

measure bidder n's strength (power) relative to bidder m at a given value v. Similarly, define

$$U_n(v) = (v - s)\prod_{m \neq n} F_m[\varphi_m(s)]$$

as bidder n's equilibrium expected pay-off (profit) at an auction if his value is v, and let

$$R_{n,m}(v) = \frac{U_n(v)}{U_m(v)},\ v \in [\underline{v}, \overline{v}]$$

denote bidder n's equilibrium pay-off relative to bidder m's equilibrium pay-off at a given value. Note that $P_{n,m}(v)$ is exogenous, while $R_{n,m}(v)$ is endogenous—we shall use this language to refer to these ratios.

Kirkegaard [31] demonstrated that the two ratios can be used to make predictions concerning the properties of $\sigma_n(v)$ and $\sigma_m(v)$ or, equivalently, $\varphi_n(s)$ and $\varphi_m(s)$. At v equal $\overline{v}$, the two bids coincide and so too do the two ratios, or $\sigma_n(\overline{v})$ equals $\sigma_m(\overline{v})$ and $R_{n,m}(\overline{v})$ equals $P_{n,m}(\overline{v})$, which is one. In fact, comparing the two ratios at any $v \in (\underline{v}, \overline{v}]$ is equivalent to comparing the equilibrium bids at v, or

$$R_{n,m}(v) \gtreqless P_{n,m}(v) \Longleftrightarrow \sigma_n(v) \gtreqless \sigma_m(v),\ \text{for } v \in (\underline{v}, \overline{v}]. \tag{28}$$

Moreover, it turns out that the motion of the endogenous ratio $R_{n,m}$, is determined by how it compares to the exogenous ratio $P_{n,m}$. Specifically,

$$R'_{n,m}(v) \gtreqless 0 \Longleftrightarrow R_{n,m}(v) \gtreqless P_{n,m}(v),\ \text{for } v \in (\underline{v}, \overline{v}]. \tag{29}$$

The standard right-boundary condition can be written in terms of these ratios as

$$R_{n,m}(\overline{v}) = P_{n,m}(\overline{v}) = 1, \tag{30}$$

while the left-boundary condition, so long as the second condition from [16] is satisfied, becomes

$$\lim_{v \to \underline{v}} R_{n,m}(v) = \frac{f_m(\underline{v})}{f_n(\underline{v})} = \lim_{v \to \underline{v}} P_{n,m}(v). \tag{31}$$

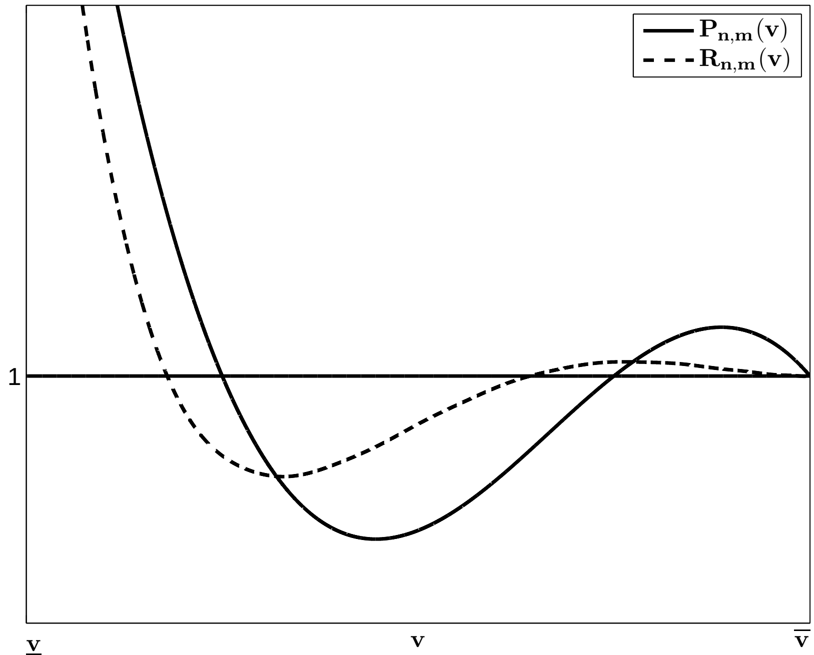

Figure 5: Comparing $R_{n,m}(v)$ and $P_{n,m}(v)$ and a path consistent with (28)–(31).

Remember: Fibich et al. maintained $f_n(\underline{v}) \in \mathbb{R}_{++}$ for all N. These observations allow one to make a number of predictions. In figure 5, we depict the exogenous and endogenous ratios for an example in which F_n and F_m cross twice in the interior, so $P_{n,m}$ equals one twice in the interior.

Based on the approximate equilibrium-bid functions, denoted $\hat{\sigma}_n$, the ratio of expected pay-offs can be computed. Denote the estimated ratio by $\hat{R}_{n,m}$. If $P_{n,m}$ and $\hat{R}_{n,m}$ are plotted in the same figure, then they should interact in a manner consistent with these observations, as illustrated in figure 5. Comparing these ratios provides a visual "test" of the adequacy of the approximation. Specifically, the steepness of $\hat{R}_{n,m}$ at a point of intersection with $P_{n,m}$ and the location of the intersections can be used to eliminate inaccurate solutions. In particular, any acceptable solution should respect the following:

1. **Slope**: At any point where $P_{n,m}$ and $\hat{R}_{n,m}$ intersect (i.e., where $\hat{\sigma}_n$ equals $\hat{\sigma}_m$), the latter should be *flat*, have a derivative that equals zero. If $\hat{R}_{n,m}$ is steep at such a point, then this is an indication that the approximate equilibrium-bid function is inaccurate as the first-order conditions are not close to being satisfied. Note, too, that this is true any time bids coincide (for any $v > \underline{v}$, including $\overline{v}$).
2. **Location**: The location of the intersections of $P_{n,m}$ and $\hat{R}_{n,m}$ must also be consistent with theory. In particular, $P_{n,m}$ and $R_{n,m}$ can cross *at most* once between any two peaks of $P_{n,m}$; when the diminishing wave property is satisfied (see Kirkegaard [31]), they *must* cross between any two peaks (not counting v equals $\overline{v}$). In figure 5, for example, $P_{n,m}$ and $R_{n,m}$ must cross once to the left of the point where $P_{n,m}$ is minimized, and once between the two interior stationary points.

Although Hubbard et al. chose to use a polynomial approximation approach, the "tests" they proposed to check the validity of a candidate solution can be used regardless of the approximation method used by a researcher.

The authors also considered a Monte Carlo study using various orders of approximations to solve some examples of asymmetric auctions. They found that poor approximations (those involving polynomials of too low order failed the visual test suggested above) led to incorrect expected-revenue rankings between first-price and second-price auctions, incorrect insights concerning the number of inefficiencies that obtain, and incorrect conclusions concerning which auction format favours different bidders as well as how auction formats affect the *ex ante* probability of a given bidder winning the auction. In short, at the risk of belaboring the obvious, it is important that researchers use good approximations.

5. Some Examples

In this section, we present the approximate solutions to some examples of equilibrium inverse-bid functions. Our approach mirrors our previous presentation: we begin by considering a problem for which we know the solution and then consider increasingly more difficult problems. Specifically, we first consider approximating the solution to a symmetric auction, but treating it like an asymmetric auction. This is something we should expect any numerical approach to do successfully; doing this also allows us to benchmark a given method since the bidding strategy can be computed in closed-form. We then consider a common example studied by economic theorists in which there are two bidders at auction and the private-values distribution of one bidder first-order stochastically dominates that of the other. In the third example, we examine a problem that, until recently, had not been investigated, one which involves value distributions that cross. Finally, in our last example, we investigate a problem that has been neglected (relative to the independence case), one in which bidders draw valuations from different marginal distributions, but (following Hubbard and Paarsch [24]), we allow these valuations to be dependent by choosing a copula that imposes affiliation.

Example 1: Consider a first-price auction with no reserve price involving two bidders who each draw valuations randomly from a standard uniform distribution. That is, $F_1(v)$ and $F_2(v)$ are both uniform distributions on the interval $[0, 1]$. In this setting, the symmetric bid function presented in equation (6) simplifies to

$$\begin{aligned}\sigma(v) &= v - \frac{\int_0^v u \, \mathrm{d}u}{v} \\ &= v - \frac{v^2}{2v} \\ &= \frac{v}{2}.\end{aligned}$$

Note that this agrees with the bid functions derived in closed-form in section 2.6 for the asymmetric uniform setting in which $\bar{v}_1$ and $\bar{v}_2$ both equal one. A useful first step in trying to approximate the (inverse-) bid functions at an asymmetric auction is to consider a symmetric auction, but treat it like an asymmetric auction by solving the appropriate system of differential equations. Regardless of the example, the approximations should all equal one another. Furthermore, they should be consistent with the closed-form bid function which can be computed for any choice

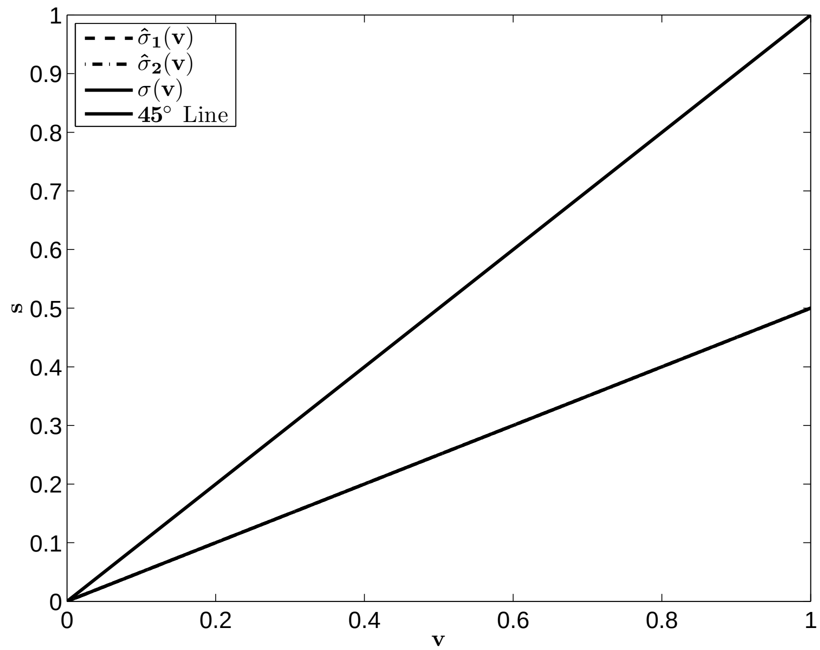

Figure 6: Closed-Form and Approximate Equilibrium-Bid Functions for Two Symmetric Uniform Bidders

of distribution. The uniform example we use is particularly attractive because the integral over the distribution of the maximum valuation of a rival bidder can be solved in closed form. Were this not the case, some kind of quadrature routine would be required. In figure 6, we depict the approximate equilibrium-bid functions as well as the true bid functions alongside the 45-degree line. Note that only two lines are clear in the figure—the 45-degree line and the closed-form bid function because the approximations match the true bid function exactly. □

Given that we have solved a symmetric auction with some confidence, we next consider a small modification to this problem. Specifically, we change the distribution from which one bidder draws valuations. A convenient setting for our examples involves the (standard) Beta distribution which has support $[0, 1]$. The Beta probability density function can be expressed as

$$f(v; \theta_1, \theta_2) = \frac{\Gamma(\theta_1 + \theta_2)}{\Gamma(\theta_1)\Gamma(\theta_2)} v^{\theta_1 - 1}(1 - v)^{\theta_2 - 1} \quad \theta_1 > 0,\ \theta_2 > 0,\ 0 \le v \le 1$$

where

$$\Gamma(x) = (x - 1)!$$

when x is an integer. The cumulative distribution function is then

$$F(v; \theta_1, \theta_2) = \frac{\Gamma(\theta_1 + \theta_2)}{\Gamma(\theta_1)\Gamma(\theta_2)} \int_0^v u^{\theta_1 - 1}(1 - u)^{\theta_2 - 1}\, \mathrm{d}u,$$

which is often referred to as the *regularized Beta function*, while the integral term is referred to as the *incomplete Beta function*. Note that, when θ_1 and θ_2 both equal one, the Beta distribution

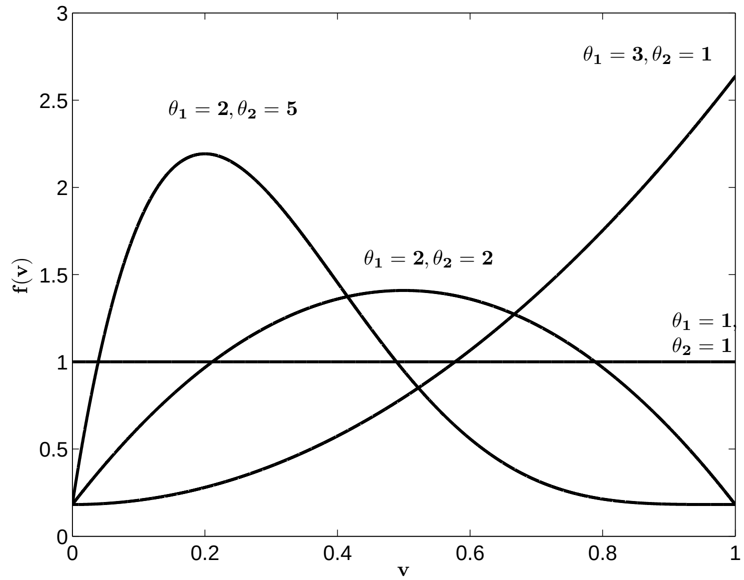

Figure 7: Beta-Uniform Mixture Probability Density Functions

is simply a uniform distribution. This distribution is attractive to use because the probability density functions can capture a wide array of shapes. Unfortunately, for many parameterizations, the probability density function takes the value zero at $\underline{v}$ equal zero. To ensure the probability density function is strictly positive, we avoid this difficulty by mixing the Beta distribution with a uniform distribution and choosing the weight on the uniform distribution to be small—to preserve the properties that make the Beta distribution attractive. We depict some Beta-Uniform mixed probability density functions in figure 7 under various parameterizations.[38] In figure 8, we depict the corresponding cumulative distribution functions as we shall reference them in the two examples that follow.

Example 2: Consider a first-price auction with no reserve price involving two bidders. Assume valuations for bidder 1 are distributed uniformly, so

$$F_1(v) = F(v; 1, 1),$$

while valuations from bidder 2 are distributed via the following Beta-Uniform mixture distribution:

$$F_2(v) = \omega F(v; 1, 1) + (1 - \omega) F(v; 3, 1)$$

with the weight ω equal 0.1. This latter distribution puts more weight on higher valuations and first-order stochastically dominates the uniform distribution as shown in figure 8—the mixture distribution involving $F(v; 3, 1)$ lies everywhere to the right of the uniform $F(v; 1, 1)$ distribution. Because of this dominance, bidder 2 is referred to as the *strong* bidder, while bidder one is the

[38] We chose the weight on the uniform distribution to be 0.1, so the weight on the Beta distribution was 0.9.

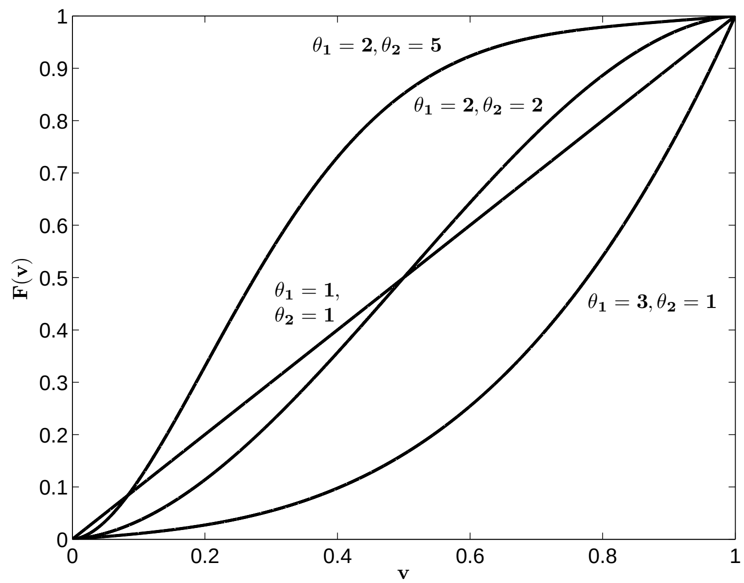

Figure 8: Beta-Uniform Mixture Cumulative Distribution Functions

weak bidder. Under such settings, Lebrun [36] as well as Maskin and Riley [43] have established that a weak bidder bids more aggressively than a strong one: weakness leads to aggression, as Krishna [33] put it.[39] We present the approximate equilibrium-bid functions in figure 9. Note that they are consistent with this theoretical insight—for any fixed valuation $v \in (\underline{v}, \overline{v})$, bidder 2 shades his bid by more than bidder 1. An implication of this is that when one distribution stochastically dominates the other, the bid functions can never cross. This is a minimal check researchers can verify for any approximations. □

Although the sharp predictions of the model with strong and weak bidders make it an attractive one, little reason exists to suggest that the model is necessarily an accurate description of real-world bidder asymmetries. As we discussed above, recently, Kirkegaard [31] derived results under much weaker assumptions concerning the primitives of the economic environment. For example, he has shown that when first-order stochastic dominance does not hold, so the cumulative distribution functions of bidders cross, the equilibrium bid functions must cross as well. In the next example, we consider a situation where the cumulative distribution functions of the bidders cross. As shown by Hubbard et al. [26], the guidance that theory provides in such settings is extremely helpful to researchers interested in approximating an asymmetric first-price auction as it allows for further qualitative "checks" on the approximate solution.

Example 3: Consider a first-price auction with no reserve price involving two bidders. Assume

[39] In fact, when deriving their results, Lebrun as well as Maskin and Riley assumed *reversed hazard-rate dominance*, which is stronger than first-order stochastic dominance: under reversed hazard-rate dominance, the ratio of the probability density function to its cumulative distribution function of the strong bidders is point-wise larger than that for the weak bidders. Krishna [33] has provided an intuitive discussion of this powerful result.

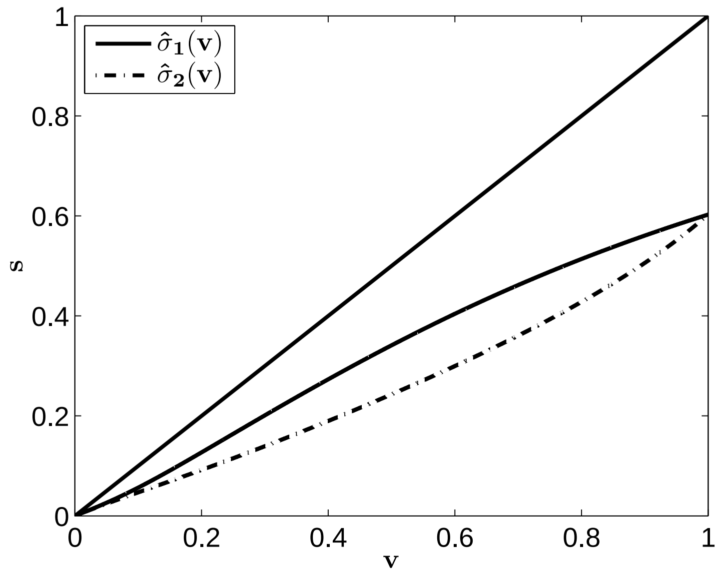

Figure 9: Approximate Equilibrium-Bid Functions at Asymmetric Auction for Example 2

valuations for bidder 1 are distributed uniformly, so

$$F_1(v) = F(v; 1, 1),$$

while valuations from bidder 2 are distributed via the following Beta-Uniform mixture distribution:

$$F_2(v) = \omega F(v; 1, 1) + (1 - \omega)F(v; 2, 2)$$

with the weight ω equal 0.1. The two distributions cross when v equals 0.5 as depicted in figure 8. Note, too, that $F_1(v)$ is a mean-preserving spread of $F_2(v)$. In figure 10, we depict the approximate equilibrium-bid functions for this example. How can we be assured that these approximations are reasonable? We follow Hubbard et al. [26] and, in figure 11, depict over a restricted interval the exogenous and endogenous ratios discussed earlier. We restrict our plot to this interval to highlight the interesting aspects of the figure. Note that $\hat{R}_{2,1}$ intersects $P_{2,1}$ once as the bid functions cross once. The crossing is at an appropriate point as it lies between $\underline{v}$ and the interior stationary point of $P_{2,1}$. When $\hat{R}_{2,1}$ is less (greater) than $P_{2,1}$, $\hat{R}_{2,1}$ is decreasing (increasing). As such, $\hat{R}_{2,1}$ is at a stationary point when it intersects $P_{2,1}$. These observations should be consistent for a given approximation for an situation in which the valuations distributions cross. □

Thus far, all of our focus (both in the examples and in our earlier presentation) has concerned the IPVP. The symmetric IPVP model involves bidders whose valuations are independent and identically-distributed. While our research has been motivated by relaxing the commonly-adopted symmetry assumption (valuations are no longer identically distributed), the methods we have discussed can also be used to examine models in which valuations are dependent as well. In

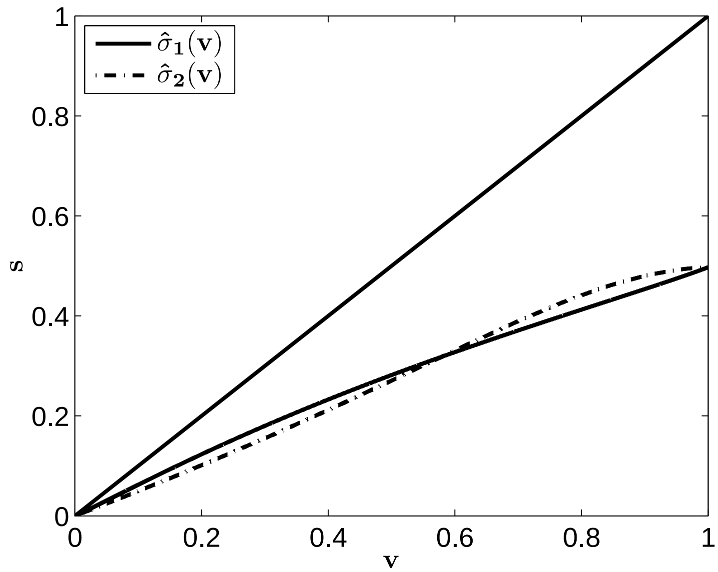

Figure 10: Approximate Equilibrium-Bid Functions at Asymmetric Auction for Example 3

auction theory, it has been typically assumed that dependence satisfies affiliation, a term coined by Milgrom and Weber [47]. Affiliation is a condition concerning the joint distribution of signals. In the case of continuous random variables, following Karlin [30], some refer to affiliation as multivariate total positivity of order two. Essentially, under affiliation for continuous random variables, the off-diagonal elements of the Hessian of the logarithm of the joint probability density of signals are all non-negative—the joint probability density function is log-supermodular. Maskin and Riley [44] showed that a monotonic equilibrium exists when bidders draw valuations from heterogenous distributions within the APVP. Hubbard, Li, and Paarsch [27] used the family of Archimedean copulae to impose affiliation in an econometric model, thus, ensuring an equilibrium is satisfied by the measurement equation. Rather than derive an entirely new model, we refer readers to Hubbard and Paarsch [24] who investigated the asymmetric APVP in detail. In particular, they solved for the equilibirum (inverse-) bid functions in various theoretical models in which valuations are affiliated. In the following example, we consider asymmetric bidders who draw valuations from a joint distribution that is characterized by a Frank copula involving positive dependence. Within the Archimedean family of copulae, conditions of the dependence parameter(s) can be used to guarantee affiliation.

Example 4: Consider a first-price auction with no reserve price involving two bidders. The bidders draw valuations from a joint distribution $F_{\mathbf{V}}(v_1, v_2)$ which has compact support $[0, 1]^2$. Appealing to Sklar's theorem, we can assert that a unique copula $C[F_1(v_1), F_2(v_2)]$ exists that binds $F_{\mathbf{V}}(v_1, v_2)$ with $F_1(v_1)$ and $F_2(v_2)$.[40] Within the Archimedean family of copulae, we consider a Frank copula with dependence parameter θ_3 set such that Kendall's τ equals 0.5, which

[40] Additional details concerning this example can be found in Hubbard and Paarsch [24].

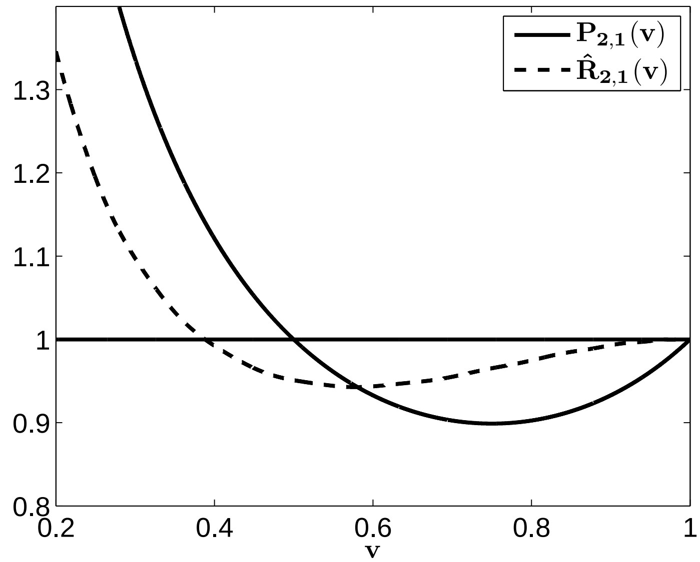

Figure 11: Exogenous and Endogenous Ratios for Example 3

implies non-negligible statistical dependence between V_1 and V_2. Assume further that valuations for bidder 1 have a uniform marginal distribution, so

$$F_1(v) = F(v; 1, 1),$$

while valuations from bidder 2 are distributed via the following Beta-Uniform mixture marginal distribution:

$$F_2(v) = \omega F(v; 1, 1) + (1 - \omega)F(v; 3, 1)$$

with the weight ω equal 0.1. Note that these are the same marginal distributions used in Example 2 which involved valuations that were independently drawn. For this example, the two relevant first-order conditions can be expressed as

$$1 = [\varphi_1(s) - s] f_2[\varphi_2(s)] \frac{\mathrm{d}\varphi_2(s)}{\mathrm{d}s} \left[\frac{\rho(s) \exp(-\theta_3 F_2[\varphi_2(s)])}{\exp(-\theta_3 F_2[\varphi_2(s)]) - 1} \right]$$

and

$$1 = [\varphi_2(s) - s] f_1[\varphi_2(s)] \frac{\mathrm{d}\varphi_1(s)}{\mathrm{d}s} \left[\frac{\rho(s) \exp(-\theta_3 F_1[\varphi_1(s)])}{\exp(-\theta_3 F_1[\varphi_1(s)]) - 1} \right]$$

where the terms in brackets at the end of each equation come from the first and second partial derivatives of the Frank copula with

$$\rho(s) = \frac{-\theta_3[\exp(-\theta_3) - 1]}{[\exp(-\theta_3) - 1] + \exp(-\theta_3 F_1[\varphi_1(s)]) \exp(-\theta_3 F_2[\varphi_2(s)])}$$

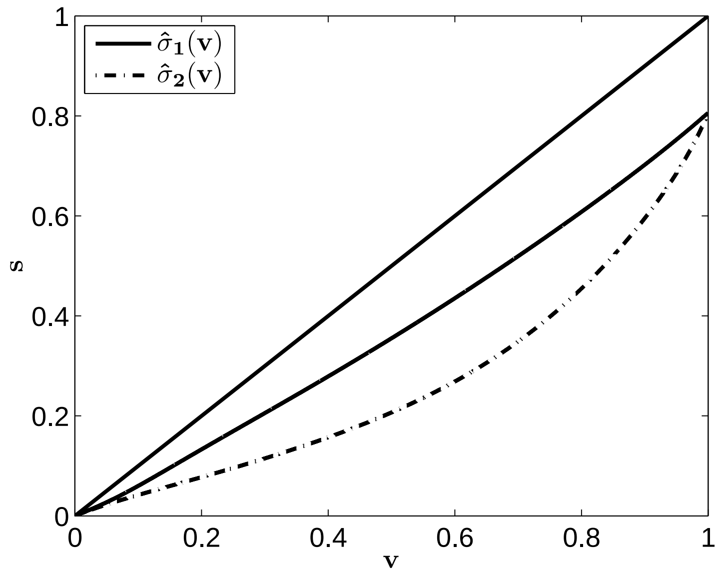

Figure 12: Approximate Equilibrium-Bid Functions at Asymmetric Auction for Example 4

which is greater than zero, provided θ_3 is greater than zero. Recall that the marginal distribution $F_2(v)$ first-order stochastically dominates $F_1(v)$ in this example. Consider the approximate equilibrium-bid functions depicted in figure 12. The weakness-leads-to-aggression result still holds, but there are clear differences in the behaviour of bidders that can be seen by comparing this figure with that of the independence case depicted in figure 9. The common high bid is much larger and the weak player bids more aggressively, especially for high valuation draws. A strong player of a given type (having a given valuation draw) often shades his bid more than in the independence case, although the bid function for the strong player increases rapidly for high valuation draws and leads to high types behaving more aggressively than under independence. Thus, both bidders behave more aggressively for sufficiently high values when there is positive dependence in the joint distribution of valuations. Affiliation disciplines bidders: when a bidder contemplates his having the highest valuation and, thus, winning the auction, he must also recognize that, under affiliation, his opponents will probably have valuations close to his, and this forces him to bid more aggressively than he would under independence, at least for bidders with sufficiently high valuations. □

6. Comparisons of Relative Performance and Potential Improvements

In this section, we do two things: first, we present a practical comparison of the shooting, fixed-point iterative, and polynomial approximation strategies focusing on run time as well as an error analysis of three of the examples described above; second, we discuss ways in which current numerical strategies could be improved. While many of these ideas are preliminary, we think they will make interesting extensions.

6.1. Comparisons of Relative Performance

Using the first three examples presented above, all of which concern the IPVP, we conducted a small error analysis of each of the three numerical strategies. For each example, we evaluated each first-order condition of the system of differential equations at a uniform grid of one thousand points. Thus, given the shooting algorithm and the polynomial approximation approach, we solved for the inverse-bid functions on the relevant grid of $[\underline{s}_m, \overline{s}_m]$ where the m subscripts denote approximated values using solution method m. For the fixed-point iterative method, on the other hand, we used one of the player's valuations as the dependent variable and solved for that player's bid function as well as the valuation(s) of the other player(s) as a function of this valuation.[41]

For each method, we considered three choices for key decision parameters that a user must choose: for the shooting algorithm, we chose three different tolerance criteria; for the fixed-point iterative method, we chose three different step sizes on the grid; for the polynomial approximation approach, we chose three different orders for the polynomials. To distinguish among these cases, in tables 1 and 2 below, we denote the shooting algorithms by "Shoot ($\varepsilon = c$)" where c denotes some tolerance criteria (at $\underline{v}$). We denote the fixed-point iterative method by "FPI ($T = x$)" where the value x represents the number of points in the uniformly-spaced grid.[42] Finally, we denote the polynomial approximation approach by "Poly ($K = d$)" where the value d represents the order of these polynomials; we present three sets of results where we varied the order of the polynomials used to approximate each inverse-bid function.

Because the iterative approach involved solving the system at a grid of points, we evaluated non-grid points using linear interpolation as this is consistent with how the algorithm operates and preserves monotonicity of the solution. Evaluating the first-order conditions required the derivatives of the inverse-bid functions; see, system (11) above for the two-player case as well as equations (19a) and (20a) in Fibich and Gavish [15]. To compute these derivatives, given a grid of points, we used finite differences and report error statistics based on the midpoint of the original grid used to evaluate the solution. This, too, is consistent with the approach of Fibich and Gavish, but in doing this we lose one grid point in the error analysis for the fixed-point iterative method which is actually based on 999 uniformly-spaced points.

We present two types of information concerning the performance of the three different numerical strategies. First, in table 1, we report the low and high bids as well as the low and high valuations under each strategy.

The lower and upper points are critical points which all strategies lever and often use as a metric to evaluate convergence. Furthermore, they provide a caveat concerning how the results of the error analysis should be interpreted because the points in the error analysis will lie within these respective intervals, a point we shall return to below. In examining table 1, the most striking result is that many of the elements concerning the shooting algorithm read "n/a": in these cases, we could not achieve convergence for the shooting algorithm. By any standard, however, very modest tolerance levels were used: our most stringent tolerance level was only 10^{-3}. Convergence could only be achieved at a tolerance of 0.1 for all examples, which is ten

[41] It is unclear how to interpret the function which maps one valuation into the other.

[42] We thank Nir Gavish for providing us with the MATLAB code used by Fibich and Gavish [15]. This code will be available on the *Games and Economic Behavior* website in due course. Aside from varying the step size of the grid and altering the distributions used, we also let the dependent variable span $[h, \overline{v}]$, where h is the step size in the grid used, rather than $[h, \overline{v} - h]$ as this seemed to improve performance at the upper end. We let all other parameters in the code take on the values recommended by the authors. In particular, the results we present are based on fifty iterations and initial guesses of $v_1 = v_2$ and $s = v_2/2$.

Table 1: Approximated Values for Extreme Points

Example	Method	$[\underline{v}_1, \overline{v}_1]$	$[\underline{v}_2, \overline{v}_2]$	$[\underline{s}, \overline{s}]$
Ex 1	Shoot ($\varepsilon = 0.1$)	[0.01082, 1.00000]	[0.01082, 1.00000]	[0.00000, 0.50016]
	Shoot ($\varepsilon = 0.01$)	[0.00283, 1.00000]	[0.00283, 1.00000]	[0.00000, 0.50020]
	Shoot ($\varepsilon = 0.001$)	[0.00001, 1.00000]	[0.00001, 1.00000]	[0.00000, 0.50062]
	FPI ($T = 500$)	[0.00200, 0.99801]	[0.00200, 1.00000]	[0.00100, 0.49950]
	FPI ($T = 5000$)	[0.00020, 0.99980]	[0.00020, 1.00000]	[0.00010, 0.49995]
	FPI ($T = 50000$)	[0.00002, 0.99998]	[0.00002, 1.00000]	[0.00001, 0.50000]
	Poly ($K = 10$)	[0.00000, 1.00000]	[0.00000, 1.00000]	[0.00000, 0.50000]
	Poly ($K = 20$)	[0.00000, 1.00000]	[0.00000, 1.00000]	[0.00000, 0.50000]
	Poly ($K = 30$)	[0.00000, 1.00000]	[0.00000, 1.00000]	[0.00000, 0.50000]
Ex 2	Shoot ($\varepsilon = 0.1$)	[0.05909, 1.00000]	[0.06997, 1.00000]	[0.00000, 0.60181]
	Shoot ($\varepsilon = 0.01$)	[0.00768, 1.00000]	[0.00776, 1.00000]	[0.00000, 0.60252]
	Shoot ($\varepsilon = 0.001$)	n/a	n/a	n/a
	FPI ($T = 500$)	[0.00200, 0.99447]	[0.00200, 1.00000]	[0.00100, 0.60103]
	FPI ($T = 5000$)	[0.00020, 0.99944]	[0.00020, 1.00000]	[0.00010, 0.60238]
	FPI ($T = 50000$)	[0.00002, 0.99994]	[0.00002, 1.00000]	[0.00001, 0.60252]
	Poly ($K = 10$)	[0.00000, 1.00000]	[0.00000, 1.00000]	[0.00000, 0.60254]
	Poly ($K = 20$)	[0.00000, 1.00000]	[0.00000, 1.00000]	[0.00000, 0.60253]
	Poly ($K = 30$)	[0.00000, 1.00000]	[0.00000, 1.00000]	[0.00000, 0.60253]
Ex 3	Shoot ($\varepsilon = 0.1$)	[0.04370, 1.00000]	[0.05997, 1.00000]	[0.00000, 0.49750]
	Shoot ($\varepsilon = 0.01$)	n/a	n/a	n/a
	Shoot ($\varepsilon = 0.001$)	n/a	n/a	n/a
	FPI ($T = 500$)	[0.00192, 0.99981]	[0.00200, 1.00000]	[0.00099, 0.49758]
	FPI ($T = 5000$)	[0.00020, 0.99998]	[0.00020, 1.00000]	[0.00010, 0.49762]
	FPI ($T = 50000$)	[0.00002, 1.00000]	[0.00002, 1.00000]	[0.00001, 0.49762]
	Poly ($K = 10$)	[0.00000, 1.00000]	[0.00000, 1.00000]	[0.00000, 0.46282]
	Poly ($K = 20$)	[0.00000, 1.00000]	[0.00000, 1.00000]	[0.00000, 0.49475]
	Poly ($K = 30$)	[0.00000, 1.00000]	[0.00000, 1.00000]	[0.00000, 0.49724]

percent of the high valuation. Because the winning bids tend to cluster at the high end, being off by ten percent at this end is particularly disappointing, particularly to those researchers hoping to use the results to inform policy debates. If the theoretical argument made by Fibich and Gavish [15] against shooting algorithms was not convincing, then this table should confirm how poorly these methods work in practice.

The fixed-point iterative method performs much better than the shooting algorithm. Remember that $[\underline{v}_2, \overline{v}_2]$ is the known interval in this approach. Unfortunately, setting $\underline{v}_2$ equal to the known value $\underline{v}$ (which equals zero in each example) prevents the algorithm from working as a singularity obtains at the low value. The solution used under this approach in practice is to simply avoid it and to start the grid a step size away from the low valuation. Thus, $\underline{v}_2$ decreases by an order as the step size increases by an order of magnitude. Unfortunately, if a researcher wanted to obtain approximations over the full true interval, then he would have to resort to extrapolation under this method.

An advantage of the polynomial approximation approach, as implemented by Hubbard et al. [26], is that theoretical constraints are imposed at the endpoints explicitly; under the polynomial

Table 2: Comparison of Algorithms Based on Run Time and Value of First-Order Conditions

Example	Method	Time (Seconds)	Value of First-Order Conditions Mean	St.Dev.	Min	Max
Ex 1	Shoot ($\varepsilon = 0.1$)	0.53591	0.00002	0.00055	0.00000	0.01726
	Shoot ($\varepsilon = 0.01$)	0.64680	0.00000	0.00001	0.00000	0.00400
	Shoot ($\varepsilon = 0.001$)	88.19402	0.00000	0.00000	0.00000	0.00001
	FPI ($T = 500$)	0.09122	0.00000	0.00000	0.00000	0.00001
	FPI ($T = 5000$)	0.22742	0.00000	0.00000	0.00000	0.00000
	FPI ($T = 50000$)	2.59620	0.00000	0.00000	0.00000	0.00000
	Poly ($K = 10$)	0.21840	0.00000	0.00000	0.00000	0.00000
	Poly ($K = 20$)	0.84241	0.00000	0.00000	0.00000	0.00000
	Poly ($K = 30$)	3.57242	0.00000	0.00000	0.00000	0.00000
Ex 2	Shoot ($\varepsilon = 0.1$)	0.87902	0.02506	0.03105	0.00000	0.11029
	Shoot ($\varepsilon = 0.01$)	1.09626	0.09558	0.13153	0.00000	0.46602
	Shoot ($\varepsilon = 0.001$)	n/a	n/a	n/a	n/a	n/a
	FPI ($T = 500$)	0.16951	0.00053	0.00066	0.00000	0.00557
	FPI ($T = 5000$)	0.61395	0.00000	0.00000	0.00000	0.00006
	FPI ($T = 50000$)	5.73171	0.00000	0.00000	0.00000	0.00003
	Poly ($K = 10$)	0.18720	0.00448	0.00308	0.00000	0.01372
	Poly ($K = 20$)	0.37440	0.00005	0.00005	0.00000	0.00036
	Poly ($K = 30$)	1.77841	0.00001	0.00001	0.00000	0.00006
Ex 3	Shoot ($\varepsilon = 0.1$)	0.44290	0.00723	0.01640	0.00000	0.35521
	Shoot ($\varepsilon = 0.01$)	n/a	n/a	n/a	n/a	n/a
	Shoot ($\varepsilon = 0.001$)	n/a	n/a	n/a	n/a	n/a
	FPI ($T = 500$)	0.17228	0.00063	0.00560	0.00000	0.23921
	FPI ($T = 5000$)	0.62627	0.00021	0.00601	0.00000	0.25841
	FPI ($T = 50000$)	5.79762	0.00020	0.00602	0.00000	0.25886
	Poly ($K = 10$)	0.18720	0.23459	0.22652	0.00000	1.97737
	Poly ($K = 20$)	0.46800	0.03214	0.04480	0.00000	0.57405
	Poly ($K = 30$)	4.18083	0.00873	0.01415	0.00000	0.22073

approximation approach, all approximations span the known interval without error for the case with uniform bidders (Example 1). Furthermore, the only unknown where error can obtain is in approximating $\bar{s}$ as it is determined endogenously. The polynomial approximation results look promising in achieving this true value: in each example, as the order of the polynomial increases the approximation of $\bar{s}$ converges in the same direction and, it appears to be the same value as the fixed-point iterative method.

In table 2, we report the time it took to solve each example for each solution method (and parameter choice) as well as some summary statistics concerning the value of the first-order conditions at the candidate solutions.

All methods do quite well in solving Example 1 for which we know the true solution. Even from this example, however, the shooting algorithm is clearly inferior: with a tolerance of 10^{-3}, the shooting algorithm took over a minute to solve, and this was the only case where this tolerance level could be attained. Remember, too, that when interpreting the error statistics, the interval over which the points were chosen might be a subset of the true interval, as shown by the results in table 1. For example, in the the application of the fixed-point iterative method to Example 1 with 500 points, the maximum error is reported as 0.00001 in table 2, but the high valuation (for bidder 1) and high bid were approximated to be 0.99801 and 0.49950, respectively, both of which involve errors that exceed the maximum reported as the true high valuation is one and the true high bid is 0.5. Note that these discrepancies have not been factored into the error statistics reported in table 2. We emphasize this caveat when interpreting the results because it can be avoided when an analytic solution exists, as in Example 1, but this is typically impossible in most cases, as the examples that follow demonstrate.

The choice of grid sizes as well as the order of the polynomials appears to be appropriate in comparing the fixed-point iterative method and polynomial approximation approach. For small grid sizes (low-order polynomials), the approximations take less than a second to calculate, while more accurate approximations (involving a finer grid or higher order polynomial) take a few seconds to solve.[43] Both methods do quite well on Example 2 which involved first-order stochastic dominance, although it is clear that the polynomial approximation approach requires a "sufficiently high" order polynomial be used, which is consistent with the arguments made by Hubbard et al. [26]. Example 3 is clearly the toughest for all approaches as the errors involved in the approximations are higher than in the first two examples. The fixed-point iterative method achieves lower error, on average, but the polynomial approximations have a lower maximum error. Note, too, that as the order of the polynomial approximation increases, all error statistics improve (the average, median, and maximum errors decrease and the standard deviation of the error gets tighter). This does not seem to hold for the fixed-point iterative method as the maximum error increases as the number of points in the grid increases (the step size decreases), the average seems to level off, and the variance in the error solution increases.

6.2. Potential Improvements

Shooting algorithms have been the most well studied and also the most criticized of the numerical strategies considered. Even in their original work, Marshall et al. [41] admitted that the analytic requirements for such shooting-based algorithms are more stringent than other numerical strategies. However, perhaps the most devastating criticism to (backwards) shooting algorithms was presented by Fibich and Gavish [15] who showed analytically that backwards shooting algorithms are inherently unstable for solving asymmetric auctions. One way to address these concerns is to incorporate checks into each candidate solution that ensure the solutions do not blow up at any point, that the inverse-bid functions are contained on $[\underline{v}, \overline{v}]$ and that they are monotonically decreasing (since we shoot backwards). Only if a candidate solution passes these checks should the convergence criteria—how close the inverse-bid functions are to $\underline{v}$, be considered. These checks require more sophisticated programming and seem absent from the methods currently proposed in the literature. Even if the instability concerns could be addressed, however, repeatedly shooting backwards is an expensive procedure in terms of time (as the above comparison suggested).

Applying fixed-point iteration (or Newton's method) as suggested by Fibich and Gavish [15] appears very promising. Although not discussed in their paper, it appears, from the authors' code that they use a Gauss–Seidel method in which, within an iteration, the updated values of solutions that have already been considered are used in computing solutions to the remaining equations.

[43] These calculations were performed on a MacBook Pro having 4GB of memory and a 2.3 GHz dual-core Intel Core i5 processor. In short, a modest computer.

This is used to speed-up convergence. There are two ways in which researchers can immediately consider improving the iterative methods. The primary concern with such an approach is that it requires a transformation of the system of differential equations which seems to depend critically on which variable (player's valuation) is chosen as the independent variable. Unfortunately, this choice is critical—the wrong choice can lead to a divergent solution. Future research in this area might suggest guidance into this choice. For example, perhaps there is a relationship between the distributions each player draws valuations from (which is the reason the asymmetry obtains) that can be used to identify which variable should be used as the independent variable.

A second area for improving the iterative approach involves deriving a rule to determine the initial guess. The importance of a good initial guess had been stressed by all researchers. For example, a classic criticism of Newton-based methods is that there is often a small region of convergence—meaning an initial guess must be quite close to the solution. In finding a zero of a function over a closed interval, researchers might choose the midpoint of the interval. Fibich and Gavish [15] chose an analogous approach in constructing an initial guess for the iterative methods for solving nonlinear systems of equations. For example, in the two-player case the authors used a symmetric bidding strategy (the valuations are equal to each other) and a uniform-based rule in which bids are one-half of the valuation considered as their initial guess. They found this worked to approximate well the solutions in the examples they considered but suggested future research might consider the sensitivity of an initial guess.

The polynomial-approximation and other approaches related to spectral methods can also be improved. While the methods are quicker than shooting algorithms, a researcher cannot use these methods blindly: solutions need to be inspected to make sure they are reasonable. For example, a common issue with Chebyshev polynomials is that they can be difficult to implement when it comes to shape constraints. To deal with this, Hubbard et al. [26] imposed rationality (players cannot bid more than their valuation) and monotonicity constraints on a finer grid of points which were not used in the residual calculation. We believe there are other ways to improve the performance of these approximation methods; the discussion that follows focuses on extensions or modifications to this approach. Specifically, we believe researchers might have success by changing the norm used in the objective, by choosing different bases or by expanding the types of terms involved in the approximation, or by considering a finite element method.

Bajari [3] originally considered a least-squares objective and Hubbard and Paarsch [23] as well as Hubbard et al. [26] continued this tradition, but some problems might be better approximated using a different choice of norm. Returning to Bajari's original (third) method, he suggested using ordinary (algebraic) polynomials. These polynomials are very similar (correlated) with each other and as such, the least-squares problem is ill-conditioned. While this basis is probably not a good choice using a least-squares approach, they might be more appropriate under a different norm. For example, rather than consider a least-squares objective, which involves a L_2 (Euclidean) norm, a researcher could minimize the sum of the absolute values of the residual function, which involves a L_1 (taxicab) norm or Manhattan distance. This will seem sensible to applied researchers that have found least-squares does not perform well in the presence of outliers and have, instead, opted to use a least absolute deviation (LAD) approach which targets the median (as opposed to the mean) and is robust to outliers. In the auction case, outliers would be like poorly approximated regions of an inverse-bid function. While an orthogonal basis has a comparative advantage in an L_2 setting, this concept concerns vector spaces and adds nothing when an L_1 norm is adopted. An advantage of casting the problem in an L_1 norm is that it allows the problem to be transformed into a linear programming problem. For example, maintaining the spirit of the approach suggested by Hubbard et al. [26], a researcher could chose the unknown parameters (the common high bid and the polynomial coefficients) to minimize

$$\sum_{n=1}^{N}\sum_{t=1}^{T}|G_n(s_t;\bar{s},\alpha)|$$

where $G_n(\cdot)$ was defined in equation (25) and the optimization is subject to the four boundary-related constraints we've discussed as well as rationality and monotonicity-related conditions. In practice, this problem can be recast as the following linear programme:

$$\min_{\{\bar{s},\alpha\}} \sum_{n=1}^{N}\sum_{t=1}^{T} E_{nt}$$

subject to

$$-E_{nt} \le G_n(s_t;\bar{s},\alpha) \le E_{nt},\; n = 1,\dots,N,\; t = 1,\dots,T$$

as well as the boundary and shape constraints.

A researcher might instead opt for a L_∞ (maximum) norm which would involve choosing the unknown parameters to minimize

$$\max\{|G_n(s_t;\bar{s},\alpha)|\}\;\; n = 1,\dots,N,\; t = 1,\dots,T,$$

subject to the boundary and shape constraints discussed. Here, the error in the approximation to the system is defined as the largest value, at any given point in the grid, that one of the first-order conditions takes. (Remember, the first-order conditions would all equal zero exactly in a perfect solution.) This strategy is often called a minimax approximation by researchers concerned with numerical analysis and this approach has been used for some time to approximate solutions to systems of ODEs (see, for example, Burton and Whyburn [7]) as well as boundary value problems (see, for example, Schmidt and Wiggins [56]). For more on the choice of norm in approximating these solutions, we refer readers to Hubbard et al. [25] who explored the use of the L_1 and L_∞ norm in solving asymmetric auctions.

Alternatively, an improved approximation could involve choosing a different basis. Judd [28] noted that if the choice of basis is good, then increasing the order of the approximation should yield better approximations. He went on to claim that "using a basis that is well-suited to a problem can greatly improve performance." See the discussion on pages 380–382 of his book. This suggests that a researcher must take care in selecting a basis. While we have advocated Chebyshev polynomials, other orthogonal basis such as Hermite, Laguerre, or Legendre polynomials may make for a better choice on a given problem. A researcher might also have success in choosing a basis which involves some terms that have characteristics that basis-members are known to have in common with the solution a researcher is trying to approximate. Judd [28] eluded to this by suggesting that one choose basis elements that look something like the solution so that few elements can give a good approximation. Outside of the class of orthogonal polynomials, there may be gains from choosing Bernstein or Laurent polynomials. The Bernstein polynomials are defined by

$$\mathbb{B}_{k,K}(s) = \binom{K}{k} s^k (1-s)^{K-k}$$

and might make for an attractive basis in this problem because the first few polynomials put weight on the approximations near the boundaries which is often where most of the concern in the approximations are. The Laurent polynomials are like standard algebraic polynomials but they allow for negative powers to be included in the approximation. This might be attractive in capturing the singularity that obtains at the boundary of these problems. Ultimately, a researcher can include a linear combination of functions from an arbitrary dictionary as projection methods generalize directly to idiosyncratic basis element choices. [58] proposed a least-absolute deviation shrinkage and selection operator (*lasso*) technique in an estimation setting. The lasso approach reduces the value of (shrinks) some coefficients like in ridge regression and, hence, is more stable, and sets other values to zero, like in subset selection, if the variable/element is not important. It would seem such an approach could be used to determine which elements of a basis are helpful in approximating the inverse-bid functions via projection methods. Related work concerning estimation of a function is the basis pursuit approach which involves minimizing an L_1 norm of the coefficients; see, for example, [10].

Rather than approximating each inverse-bid function by one global function (polynomial), one might consider finite-element methods which involve partitioning the domain of interest ($[\underline{v}, \bar{s}]$ in our case) into smaller segments and using splines or piecewise polynomials which are then aggregated to form an approximation over the entire region. This is like an hybrid of the polynomial approach originally suggested by Bajari [3] that comes closer to capturing the spirit of the Taylor-series expansions suggested by, most recently, Gayle and Richard [18]. However, this approach avoids the use of backwards integration and would not suffer from the criticisms of [15] as it involves a nonlinear least-squares (or LAD, depending on the norm) approach. This might be attractive as it can allow the researcher to gain the speed of polynomial approach but perhaps reduce error and improve stability.

7. Summary and Conclusions

In this paper, we have presented a survey of numerical methods used to approximate equilibrium bid functions in models of auctions as games of incomplete information where private values are modelled as draws from bidder-specific type distributions when pay-your-bid rules are used to determine transaction prices. We first outlined a baseline model and demonstrated how things work within a well-understood environment. Subsequently, we described some well-known numerical methods that have been used to solve two-point boundary-value problems that are similar to ones researchers face when investigating asymmetric first-price auctions. Next, we discussed research that either directly or indirectly contributed to improving computational methods to solve for bidding strategies at asymmetric first-price auctions. We also depicted the solutions to some examples of asymmetric first-price auctions to illustrate how the numerical methods can be used to investigate problems that would be difficult to analyze analytically. In fact, we presented a solution to one example that has received very little attention thus far—asymmetric auctions within the APVP. We also compared and contrasted the established methods and suggested ways in which they can be extended or improved by additional future research. Finally, by providing the computer code used to solve the examples of asymmetric first-price auctions, we hope to encourage researchers to apply these methods in their research.

A weakness of all research in this literature is that all evidence concerning the performance of the proposed approach is purely numerical and done via example—no one has considered analytically the efficiency and/or convergence properties of the proposed solutions. This shows value of work like [26] which brings in sound theoretical results to verify the solutions. Nonetheless, the shortcoming of this field is that no one has proved that an approach converges to the truth.

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